PHYS 150 TERM PAPER - YOUNG TABLEAUX AND ITS APPLICATIONS

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1. Introduction

Young tableau is a combinatorial object that is often useful in dealing with permutations, representations, and degeneracies. In this paper, we will first introduce the basic concept of Young diagram, tableau,and hook. In the later section, we will extend the theory of Young tableau and explore its relation with involution through the Robinson-Schensted-Knuth (RKS) correspondence, which will also lead us into a bit of discussion on symmetric functions and the Cauchy identity. In the last section, we will introduce some applications of the Young tableau in data structure, quantum mechanics, and quantum error correction.

2. Basic Concepts

2.1. **Partition and Young Diagram.** First of all, let us consider a partition λ on a set of n elements, which can be written as an ordered set of numbers $\{\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_m \geq 0\}$ and whose size $|\lambda| \equiv \sum_i \lambda_i = n$. To visualize this partition, we can use a Young diagram, a set of left justified cells with k rows, each ith row with λ_i cells, where λ_i is weakly decreasing as we go down the rows. An example of the partition $\{4, 2, 2, 1\}$ is shown below in **Figure 1**.

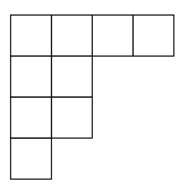

Figure 1. Young diagram for partition $\{4, 2, 2, 1\}$

A Young tableau is a Young diagram with numbers within each cell. A semi-standard Young tableau is one whose number in the cells are weakly decreasing to the right to the bottom. A semi-standard tableau of size $|\lambda| = n$ is considered standard if its filling is a bijective assignment from $\{1, 2, 3...n\}$. Example of both cases are shown below for partition $\{4, 2, 2, 1\}$.

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1	2	2	3
3	5		
5	7		
9			

(A)

1	3	5	7
2	4		
6	9		
8			

(B)

FIGURE 2. (A). Semi-standard Young tableau for partition $\{4, 2, 2, 1\}$; (B). Standard Young tableau for partition $\{4, 2, 2, 1\}$ with $|\lambda| = n = 9$

2.2. **Hook Length Formula.**

In this subsection we will introduce the idea of a hook in a Young diagram, and discuss the non-trivial hook length formula, which is useful in finding the total number of standard tableaux for any given shape, which turns out to be useful in the later applications.

Definition 2.1. A hook $H(i, j)$ is a set of cells to the right or below the cell at ith row and jth column, plus the (i, j) cell itself.

Definition 2.2. Hook length $h_\lambda(i, j)$ of a Young diagram of shape λ is the number cells in the hook $H_\lambda(i, j)$.

7	5	2	1
4	2		
3	1		
1			

FIGURE 3. A tableaux of $\lambda = \{4, 2, 2, 1\}$ filled by each cell's hook length

With the hook length defined, we can find the total number of standard tableaux of a given shape λ. There are a total of $n!$ ways to place the $\{1, 2, 3...n\}$ in each cell. Since the tableau is standard, we know that for each possible filling, the number in cell (i, j) cannot be permuted to the right or below, thereby decreasing the multiplicity by $h_\lambda(i, j)$, and we have arrived at the hook length formula, giving the total number of standard tableaux d_λ in shape λ:

$$d_\lambda = \frac{|\lambda|!}{\prod_{i,j} h_\lambda(i, j)} \tag{2.1}$$

The usefulness of this is not immediately clear, but in the following section we will go down the rabbit hole to explore the fundamental significance of hook length formula in studying permutation groups, involutions, RSK correspondence, and symmetric functions.

3. THEORETICAL EXTENSION

3.1. **Involution and the RSK Correspondence.**

In this subsection we will work with the $S4$ group as an example and proceed to show the bijective relation between number of involutions on a permutation group Sn and the total number of standard tableaux we can form from $\{1, 2, 3...n\}$.

Definition 3.1. An involution is a function, transformation, or operator whose inverse is itself.

The total number of involution in a symmetric group is given by the number of tableaux we can form. Consider the $S4$ group as an example. To construct the diagram, we can count its equivalence classes:

$(x)(x)(x)(x)$	1+1+1+1
$(xx)(x)(x)$	2+1+1
$(xx)(xx)$	2+2
$(xxx)(x)$	3+1
$(xxxx)$	4

Each class will be represented by a diagram of shape λ_k. Let α_m represent the number of elements in an $m - cycle$ in this class. Then for a given equivalence class k, we can find its partition elements by $\lambda_{ki} = \sum_{m=i}^{n} \alpha_m$. As an example, the class $(xxx)(x)$ has $\alpha_1 = 1$ and $\alpha_3 = 1$, so we can obtain a partition of $\lambda_{k=3+1} = \{2, 1, 1, 0\}$, which corresponds to the second last tableau in the figure below, along with all tableaux we can form with the S_4 group.

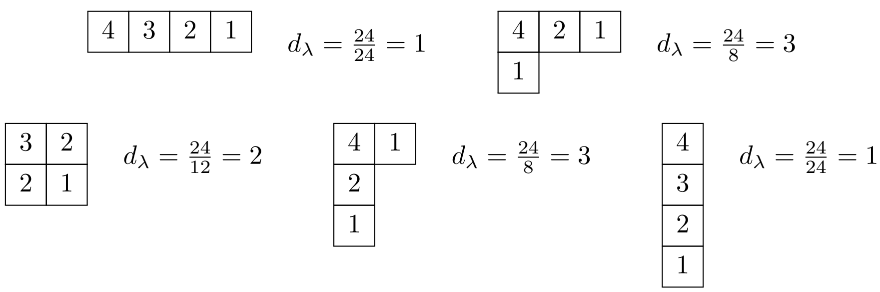


FIGURE 4. All possible tableaux for S_4 group, filled with each cell's hook length and labeled with the total number of standard Young tableaux for each given shape

Here we could use the hook length formula to find the number of elements in each equivalence class (i.e. the number of standard tableaux for each given shape). Adding up d_λ from each shape, there are a total of 10 standard tableaux we can form from the S_4 group.

Examining the resulting standard tableaux for each Young diagram, we observe that each column with n-cells represents an n-cycle in that equivalence class. Also, since we require that the order is strictly increasing in the vertical direction, it eliminates the possibility of repetitive representation of a same n-cycle. Consequently, as we fill in numbers to obtain our standard tableaux for a shape λ, we would reveal a special subset of permutations from the symmetry group, which is the involutions.

As we see there are 10 standard tableaux we can form with the S_4 group, so there are 10 involutions, as listed below:

$$\left\{\begin{matrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{matrix}\right\} \left\{\begin{matrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{matrix}\right\} \left\{\begin{matrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{matrix}\right\} \left\{\begin{matrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 1 \end{matrix}\right\}$$

$$\left\{\begin{matrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{matrix}\right\} \left\{\begin{matrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{matrix}\right\} \left\{\begin{matrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{matrix}\right\} \left\{\begin{matrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{matrix}\right\}$$

$$\left\{\begin{matrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{matrix}\right\} \left\{\begin{matrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{matrix}\right\}$$

This bijective relation is not obvious. However, this conclusion may arise naturally if we first explore the RKS correspondence, one that we are bound to mention if ever make a discussion on Young tableaux.

Theorem 3.2. *The RSK correspondence states that there exists an injective relation between the set of all permutations on* $\{1, 2, 3...n\}$ *and the the set of ordered pairs of* (P, Q)*, where* P *and* Q *are the tableaux of same shape formed from* $\{1, 2, 3...n\}$*.*

Proof. Let us have two rows of array:

$$\begin{pmatrix} p_1 & p_2 & ... & p_n \\ q_1 & q_2 & ... & q_n \end{pmatrix}, \qquad \begin{matrix} p_1 < p_2 < ... < p_n \\ q_1, q_2, ..., q_n \text{ are distinct positive integers} \end{matrix} \tag{3.1}$$

Notice that this is a more general case than what we intend to prove, which in fact allows more convenience in proof writing.

Next we will construct the two tableaux P and Q of same shape. Let $\{p_1, p_2, ...p_n\}$ be elements of P and $\{q_1, q_2, ..., q_n\}$ be elemets of Q. But before we construct them, let us first show what does it mean by tableau insertion, which we will present an example to give some intuition, and the full algorithm will be found in the appendix.

Consider the following array:

$$\begin{pmatrix} 1 & 1 & 2 & 2 & 2 & 3 \\ 3 & 4 & 1 & 1 & 2 & 1 \end{pmatrix} \tag{3.2}$$

The basic idea with tableau insertion is basically trying to insert it into the first row if possible, and then bump the original cell to the next row to let it slide down to the right position according to its value. Continue this step until we have no more cells that we have to bump.

With this basic intuition in mind, we will then use the tableau insertion algorithm to construct P out of the first row, and Q out of the second row.

$$\emptyset \quad \emptyset \quad \leftarrow 1 \quad 3 \tag{3.3}$$

$$\begin{array}{|c|}\hline 1 \\ \hline\end{array} \quad \begin{array}{|c|}\hline 3 \\ \hline\end{array} \leftarrow 1 \quad 4 \tag{3.4}$$

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 3 & 4 \\ \hline\end{array} \leftarrow 2 \quad 1 \tag{3.5}$$

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 3 \\ \cline{1-1}\end{array} \leftarrow 2 \quad 1 \tag{3.6}$$

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array} \leftarrow 2 \quad 2 \tag{3.7}$$

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 2 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 3 & 4 \\ \cline{1-2}\end{array} \leftarrow 3 \quad 1 \tag{3.8}$$

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 2 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 1 & 4 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array} \tag{3.9}$$

Here, we have the two resulting tableau P and Q above with the same shape. The tableau insertion is the inverse of the tableau deletion, so we can reverse the above process, and this will conclude the one-to-one correspondence between permutation and (P, Q). □

Now that this has been proven, we are ready to show the bijective relation between involution and standard tableau.

Corollary 3.3. *There exists a bijective relation between the set of involutions in a permutation group S_n and the set of standard tableaux formed from $\{1, 2, 3...n\}$.*

Proof. Let ξ be the involution corresponding to (P, Q), then ξ^{-1} corresponds to (Q, P) due to how we construct P and Q. Since $\xi = \xi^{-1}$, we can conclude that $(P, Q) = (Q, P)$, and that $P = Q$. So ξ correspond to (P, P).

Conversely, if we have ξ being any permutation that corresponds to (P, P), then (P, P) also corresponds to ξ^{-1}, and $\xi = \xi^{-1}$ is an involution. □

Now we have unveiled the mystery of involution and standard tableau, and we just showed that the idea of hook is non-trivial in this application, since the total number of standard tableau of a given shape can be given by the hook length formula. In exploring this relation, we found out about the RSK correspondence, which is in fact a much stronger statement than just proving the above corollary. So in the next subsection, we shall see that it presents a direct bijective proof for the Cauchy identity in symmetric functions.

3.2. **Cauchy Identity and Symmetric Functions.** First, let us give a formal definition for a symmetric polynomial.

Definition 3.4. A polynomial is symmetric if it is invariant under action of S_n.

For example, if we have $\sigma \in S_n$ acting on a polynomial $P(x_a, x_2, ..., x_n)$:

$$\sigma P(x_1, x_2, ...x_n) = P(x_{\sigma_1}, x_{\sigma_2}, ..., x_{\sigma_n}) \tag{3.10}$$

If we have $\sigma = (1 \quad 3 \quad 2)$ acting on $P = x_1x_2 + x_2^2 + x_2x_3$ then we will get $\sigma P = x_3x_2 + x_2^2 + x_2x_1$, which is just P, so we see that P is a symmetric polynomial.

Definition 3.5. The evaluation of a tableau T is$(m_1, m_2, ...)$ where m_i is the number of occurrence of i in T.

Therefore, if we have the evaluation of a tableau being 1^n, then it will just represent a standard tableau, since this will indicate that every number will show up once in the tableau.

Definition 3.6. A Schur function s_λ is defined as:

$$s_\lambda = \sum_{T(\lambda)} x^{ev(T)} \tag{3.11}$$

where $T(\lambda)$ is a semi-standard tableau of shape λ. Let us see an example where $\lambda = (2, 1)$. From this we can construct eight different semi-standard tableau.

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$$

And their evaluations are:

$$(2, 1, 0) \quad (2, 0, 1) \quad (1, 2, 0) \quad (1, 0, 2) \quad (0, 2, 1) \quad (0, 1, 2) \quad (1, 1, 1) \quad (1, 1, 1)$$

It is also not surprising to see that we have 2 standard tableaux out of this shape. And we are ready to write our Schur function for this shape:

$$s_{\lambda=(2,1)} = x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 + 2x_1x_2x_3 \tag{3.12}$$

Theorem 3.7. *(Cauchy Identity)*

$$\prod_{i,j} \frac{1}{1 - x_iy_j} = \sum_\lambda s_\lambda(x)s_\lambda(y) \tag{3.13}$$

Proof. First, let us expand the LHS with geometric series:

$$\prod_{i,j} \frac{1}{1 - x_iy_j} = \prod_{i,j} \sum_{a_{ij} \geq 0} (x_iy_j)^{a_{ij}} \tag{3.14}$$

$$= (\sum_{a_{1,1}} (x_1y_1)^{a_{1,1}})...(\sum_{a_{2,1}} (x_2y_1)^{a_{2,1}})... \tag{3.15}$$

Let us have a_{ij} correspond to a pair of tableau (P, Q) under the RSK correspondence, then we can define $C_i = \sum_i a_{ij}$, which is the number of i's in P and $R_j = \sum_j a_{ij}$ being number of j's in Q. Now, we can re-write the LHS further:

$$\sum_{a_{ij} \geq 0} \prod_{i,j} (x_iy_j)^{a_{ij}} = \sum_{a_{ij} \geq 0} (\prod_i x_i^{C_i} \prod_j y_j^{R_j}) \tag{3.16}$$

$$= \sum_{\lambda(P)=\lambda(Q)} x^P y^Q \tag{3.17}$$

$$= \sum_{\lambda(P)=\lambda(Q)} (\sum_{P(\lambda)} x^{ev(P)} \sum_{Q(\lambda)} y^{ev(Q)}) \tag{3.18}$$

$$= \sum_\lambda s_\lambda(x)s_\lambda(y) \tag{3.19}$$

□

And this is the celebrated Cauchy (or Cauchy-Littlewood) identity that is interesting to study in the context of symmetric functions. And we will call this an appropriate end for poking around from the theoretical perspective. Next, we will take a plunge into the application of tableau, with a focus on how it might be useful for physicists.

4. Applications

In this section, we will introduce some of the interesting application of the Young tableaux in various field. Before we delve into the complexity of each subfield, as a warm up, let us give a quick application of it in data structure, for those who have some interest in computer science.

Consider a binary tree of size n, that is, the binary tree has n nodes. It has a bijective relation with a stack permutation on a sequence $a_1 a_2 a_3 ... a_{2n}$ where first half are S, push, and the second half are X, pop. A valid sequence can be written out as a standard Young tableau of shape $\{n, n\}$, where the first row records all the indices of S, and the second row records all the indices of X. As a result, the set of all stack permutations on such sequence can be represented by the set of all n-node binary trees we can construct. And the total number of n-node binary trees can be found be our familiar hook length formula for a shape of $\{n, n\}$.

$$d_{\lambda=(\{n,n\}} = \frac{(2n)!}{(n+1)!n!} \tag{4.1}$$

because the first row has hook length starting from $n+1$ and goes to 2, and the second row has hook length starting from n and goes to 1.

Example 4.1. Consider a stack permutation of size 12:

$$SSSXXSSXXXSX \tag{4.2}$$

This permutes $(1, 2, 3, 4, 5, 6)$ to $(3, 2, 5, 4, 1, 6)$, and can be represented by the following Young tableau:

1	2	3	6	7	11
4	5	8	9	10	12

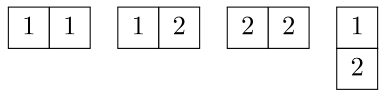
4.1. **Identical Particles.** Unlike a classical system where we can keep track of every individual particles, there exists two truly indistinguishable particles when we are dealing with a quantum system. That is, for a boson, there exists symmetry of state

$$\psi = \frac{1}{\sqrt{2}}(\psi_1(x_1)\psi(x_2) + \psi_1(x_2)\psi_2(x_1)) \tag{4.3}$$

on which if we apply the spatial exchange operator on the state by swapping the x_1 and x_2, we would get the exact same state. This means that, in this quantum system of two particles, if they secretly swapped position, there is no way we would be able to tell, not even theoretically.

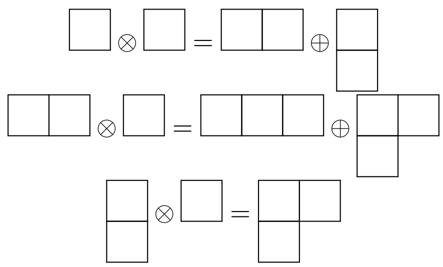
Now we can define $\boxed{1}$ to be spin up and $\boxed{2}$ to be spin down, where each single tableau cell represent a doublet. We define $\square\square$ to be symmetric and $\begin{array}{c}\square\\\square\end{array}$ to be antisymmetric.

A first look at these two diagrams we should realize that the horizontal one represent a triplet while the vertical one represent a singlet due to the amount of standard tableaux we could construct out of these shapes.



With this as a tool, we are already good to do something interesting. First, we need to keep in mind that we can never have vertical tableau of more than 2 for fermions, since total antisymmetry for more than 2 particles does not exist.

In this case, we can try the following:



We see that in the above diagrams, the problem of moving from left side to the right side simply becomes how to put the shapes together such that the criteria of antisymmetry (vertical no more than 2 cells) can be satisfied. This can then be used to deal with angular momentum addition problems since the left side is basically the tensor product of two quantum states, and the right side tells us what are the resulting angular momenta are allowed for this coupled state.

4.2. **Quantum Error Correction - Dimension of the Decoherence-Free Subspaces.** A quantum system will inevitably suffer from the error induced by decoherence, which will result in loss of information. However, we may encode the information into a subspace to protect the information over time. To give some intuition about this, let us first consider a classical case.

Suppose we have 2 coins, and we want to store 1 bit of information. This is easy. We have 2 bits of storage. However, now let us imagine a demon that would sometimes suddenly flip the sides of the coins, but fortunately, it will do the flipping on both coins simultaneously.

To still keep our information, let us redefine our bits. Let even parity be 0 and odd parity be 1. Due to the symmetry of the demon's action of these two coins, we know that the parity of the two coins would not change no matter how the demon interferes with our system. This is an example of encoding our information into a decoherence-free subspace (DFS). The advantage of it is that we do not need to recover our information. We don't need to know which bit got flipped, and we we don't need to correct for the error.

To proceed, let us first present the formal definition of a DFS to get a mathematical sense of how it proves useful in dealing with the time evolution of a quantum system.

Definition 4.2. Let there be a quantum system with Hilbert space $\mathcal{H}_S$, we call a subspace $\tilde{\mathcal{H}}_S \subset \mathcal{H}_S$ a DFS if all states $\rho_S(t=0)$ initially prepared in $\tilde{\mathcal{H}}_S$ are unitarily related to the final state $\rho_S(t)$:

$$\rho_S(t) = U_S \rho_S(t=0) U_S^\dagger \tag{4.4}$$

where $U_S : \mathcal{H}_S \to \mathcal{H}_S$ is a unitary operator.

Now consider a system S coupled to a bath B such that the Hamiltonian is $H = H_S + H_{SB} + H_B$, where $H_{SB} = \sum_\alpha S_\alpha \otimes B_\alpha$ is the interaction term, and is the reason of decoherence in this formulation. Naturally, if we have a subspace of the full system over which $H_{SB} = 0$, then in this subspace, we basically "switched off" the interaction between system and bath, and they would evolve unitarily and independently. However, this condition is too strong. $H_{SB} \propto I$ in this subspace would be good enough for what we ask for. And we could re-express this condition to explicitly show its connection to degeneracy and symmetry.

Let there be a set of eigenvectors $\{\left|\tilde{k_\alpha}\right\rangle\}$ of S_α, and we could capture the above condition by:

$$S_\alpha \left|\tilde{k_\alpha}\right\rangle = c_\alpha \left|\tilde{k_\alpha}\right\rangle \tag{4.5}$$

where c_α is the proportional constant of the condition $H_{SB} \propto I$, as we have c_α only dependent on the index of the system operator instead of the state index k_α. Essentially, for each system operator S_α, there is only one eigenvalue, and its degeneracy is the dimension of the space spanned by $\left|\tilde{k_\alpha}\right\rangle$.

To satisfy the definition of a DFS, we also need to ensure that the state $\rho(t = 0)$ stays in the $\mathcal{H}_S$, meaning that not only will the interaction term H_{SB} cause damage to our protected state, the H_S term might also take the state outside of the subspace. Therefore, we require that H_S will leave the $\tilde{\mathcal{H}}_{\mathcal{S}}$ space invariant, i.e.

$$H_S \left|\tilde{\psi}\right\rangle \in \tilde{\mathcal{H}}_{\mathcal{S}}, \forall \left|\tilde{\psi}\right\rangle \in \tilde{\mathcal{H}}_{\mathcal{S}} \tag{4.6}$$

Therefore, if we consider a system S coupled to a bath B and subject to Hamiltonian $H = H_S + H_{SB} + H_B$, where $H_{SB} = \sum_\alpha S_\alpha \otimes B_\alpha$, and the initial states of system and bath are independent of each other (i.e. $\rho_{SB}(0) = \rho_S(0) \otimes \rho_B(0)$) with condition 4.5 and 4.6 are satisfied, then there exists a DFS for the system.

And because we are dealing with qubits, the singlet states of $sl(2)$ can be represented by tableau of $\lambda = \{N/2, N/2\}$ where N is the number of qubits in this system. The multiplicity of such states can then be quickly found by the number of standard tableaux with our old friend the hook length formula. We recall that this is a familiar situation, just like when we were discussing the binary trees. With a given N, we can represent tree size of N with tableaux of $\lambda = \{N/2, N/2\}$ just like the singlet states. This gives us one more way to deal visualize the states in a DFS other than just the tableau, and pose an potentially interesting connection between data structure and representation of quantum states.

And to conclude, the dimension of a DFS is nothing more than our familiar:

$$d_{\lambda=(\{N/2,N/2\}} = \frac{(N)!}{(N/2+1)!(N/2)!} \tag{4.7}$$

5. Acknowledgement

I would like to thank Prof. Zee for assigning this term paper. I felt that the amount of math I learnt and touched on through writing this paper has exceeded the amount of math I learnt in last two years combined. It was a tremendous experience, and taught me a new way of self learning materials that are beyond the classroom level.

6. Appendix

Algorithm 6.1. (Tableau Insertion) Let P be a tableau and x be an element that does not belong to P. By insertion of x into P, we should obtain a new tableau of same shape except with a new cell in row s and column t to be determined by this algorithm. Also, let us think of the tableau to be bordered by 0's to the top and left of the cells, and by ∞'s to the right and bottom of the cells.

Step I: (Input x) Set $i = 1$, set $x_1 = x$, and set j to the smallest value such that $P_{1j} = \infty$, which means find the rightmost cell in the first row.

Step II: (Now we should have $P_{i-1,j} < x_i < P_{ij}$ and $x_i \notin P$) If $x_i < P_{i(j-1)}$ then decrease j by 1 and repeat this step to find the suitable position for x_i in this row. Otherwise, set $x_{i+1} = P_{ij}$

Step III: (Now we have $P_{i,j-1} < x_i < x_{i+1} = P_{ij}$) Set $P_{ij=x_i}$ and actually insert into the tableau.

Step IV: If $x_i+1 \neq \infty$, increase i by 1 and return to **Step II**. We will repeat the process of insertion until the last bumped cell finds the suitable position and stop bumping other cells to the next row.

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