Canonical Formulation of General Relativity

Nadie Yiluo LiTenn

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1 Introduction

In general, we approach GR through the Lagrangian formulation and variational principle, which is in any standard introductory GR textbook. Now, physically, we know that GR must be able to account for a wide range of physical phenomena, and should therefore allow for a correspondingly wide class of solutions. Among those, there exists a class for the initial value problems, which we have encountered many times in classical physics. A simple case will be the classical two-body problem, in which the motion is completely determined if we are given proper initial values. Another similar example is in electromagnetism, where the solution is completely determined once we obtain the initial value of the fields and have them satisfying the *constraints*:

$$\boldsymbol{\nabla} \cdot \vec{E} = 4\pi\rho \tag{1}$$

$$\boldsymbol{\nabla} \cdot \vec{B} = 0 \tag{2}$$

We shall then postulate that gravity shouldn't be so drastically different from other classical theories, so other than the Lagrangian formulation, it should also have an initial value formulation, perhaps subject to some constraints like electromagnetism.

1.1 A Note on Convention

In this paper, we will write quantities in component and abstract form interchageably. It should be clear from the context which form we are using. The Greek indices are take to range from 0 to 3, while the Latin indices from 1 to 3. We will also work in the so called absolute units where $c = \hbar = 8\pi G_N = 1$

We will use the $\longrightarrow$ to denote mapping between a set of objects and the $\longmapsto$ to denote the mapping of individual object in the set.

In this paper, we will denote the Riemann tensor as *Riem* and Ricci scalar as R. There should be no mixing of the two, regardless of context.

1.2 Initial Value Formulation

We shall start by a definition for the initial value formulation (IVF):

Definition 1.1. *A theory is said to possess an IVF if it can be formulated so that "appropriate initial data" may be specified (possibly subject to constraints) such that subsequent dynamical evolution of the system is uniquely determined.*

Note that this definition alone is not enough for a physically viable theory. To ensure physical viability, we further require that it is *well posed* by the following definition.

Definition 1.2. *A theory with IVF is considered well posed if it satisfies two points. First, in some appropriate sense, "small changes" on the initial data should only correspond to "small" changes on the solution over any fixed compact region. Second, changes in initial data in region S should not affect regions outside the causal future $J^+(S)$ of S.*

The above two definitions will be our central guideline for the first part of this paper to determine whether we can formulated a *well posed* IVF for a physical theory, including gravity. Now to borrow a bit more intuition from the simplest classical theory we know,

$$\vec{F} = m\vec{a} \tag{3}$$

which we can make into a more general form,

$$\frac{\mathrm{d}^2 q_i}{\mathrm{d}t^2} = F^i(q_1, ..., q_j; \mathrm{d}q_1/\mathrm{d}t\ , ..., \mathrm{d}q_j/\mathrm{d}t) \tag{4}$$

We have dealt with this theory a lot in mechanics class, where the motion of the particle is completely determined upon given initial value. Now to promote the dynamical variables to fields, and to include their coordinate labels on RHS of (4), we are expecting the new theory, written in the above form, capable of inheriting the IVF in some sense. To make things more covariant, we may want to consider further a theory that also allows dependence on $\frac{\partial^2}{\partial x^a \partial t}$ and $\frac{\partial^2}{\partial x^a \partial x^b}$. So we are essentially looking at something of the form:

$$\frac{\partial^2 \phi_i}{\partial t^2} = F_i(t, x^a; \phi_j; \partial\phi_j/\partial t\ , \partial\phi_j/\partial x^a\ ; \partial^2\phi_j/\partial t \partial x^a\ , \partial^2\phi/\partial x^a \partial x^b) \tag{5}$$

We should also note that (5) has different mathematical structure from (4). Nevertheless, we have a theorem proven by Cauchy and Kowalewski

Theorem 1.3 (Cauchy-Kowalewski)**.** *In (5), we have $t, ..., x^{m-1}$ as coordinate in $\mathbb{R}^m$, $\phi_0, ..., \phi_n$ as n unknown equations in $\mathbb{R}^m$, and F_i as analytic function of its variables. Let $f_i(x^a)$ and $g_i(x^a)$ be analytic functions. Then on the $t = t_0$ hypersurface, we have an open neighborhood $\mathcal{O}$ such that within $\mathcal{O}$, there exists a unique analytic solution to (5) where $\phi_i(t_0, x^a) = f_i(x^a)$ and $\partial\phi_i/\partial t = g_i(x^a)$.*

Essentially if we fix some analytic initial data on the $t = t_0$ hypersurface, then we have unique analytic solution, at least in some open neighborhood $\mathcal{O}$ on the hypersurface. Now we recognize that the Klein-Gordon theory (KG for short), another simple beloved theory, has exactly this form as in (5), and so must possess an IVF, at least for *analytic* initial data.

1.3 Klein-Gordon Theory

The above result shown for KG is quite subtle. It is only for *analytic* initial data, so in fact Cauchy-Kowalewski does not say anything if KG is *well posed*, that is, it is missing in establishing: 1). the continuous dependence of solution on the initial data in an appropriate sense, and 2). causal propogation of the solution.

For the first issue, Cauchy-Kowalewski cannot guarantee that, given any reasonable choice of topology on space of solution and initial data, the map taking analytic initial data to the analytic solution is continuous.

For the second issue, we know that the analytic inital surface Σ_0 can be uniquely determined if we can fix the value of a point $p \in \Sigma_0$ and the values of all the higher derivatives in a small open region U around p on Σ_0. So if we alter the initial data in a small region U, we are essentially altering all the initial data on Σ_0. In fact, this tells us that we should use non-analytic initial data instead.

So we must consider something else for KG. Let us first write down our KG equation for some massive scalar field ϕ:

$$\partial^\mu \partial_\mu \phi - m^2 \phi = 0 \tag{6}$$

The stress energy tensor associated with this theory is:

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} \eta_{\mu\nu} (\partial_\alpha \phi \partial^\alpha \phi + m^2 \phi^2) \tag{7}$$

And this is conserved.

$$\partial^\mu T_{\mu\nu} = 0 \tag{8}$$

Let the time translation Killing Vector Field be $\xi^\mu = (\partial / \partial t)^\mu$ orthogonal to the initial data surface Σ_0. We can further write

$$\partial^\mu (T_{\mu\nu} \xi^\nu) = 0 \tag{9}$$

Next, we would like to use Gauss's law for some convenient bulk and corresponding boundary (see Figure 1). Let us consider a closed ball S_0 on the initial hypersurface Σ_0 at $t = t_0$, a hypersurface Σ_1 at $t = t_1 > t_0$, a region $S_1 = D^+(S_0) \cap \Sigma_1$, a bulk region K that is sandwiched by S_0 and S_1, described by $K = D^+(S_0) \cap J^-(S_1)$, and finally S_2, the "null portion" of the boundary of K.

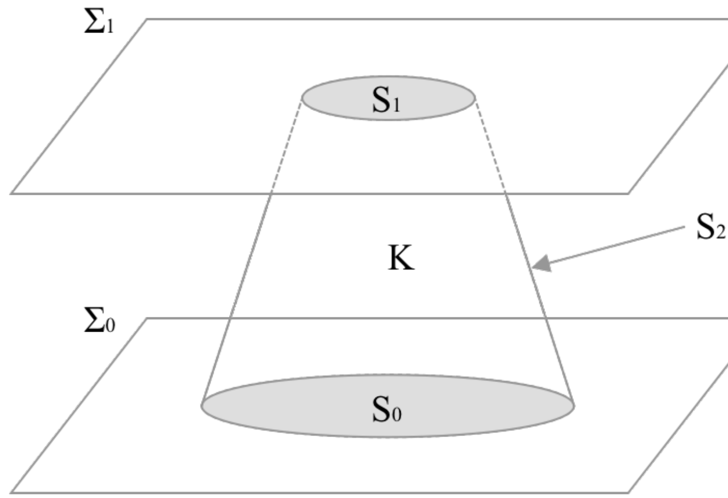

Figure 1: The convenience bulk and boundary we set up for KG.

We can then intergrate (9) over K and apply Gauss's law to obtain

$$\int_{S_1} T_{\mu\nu}\xi^\mu\xi^\nu + \int_{S_2} T_{\mu\nu}l^\mu\xi^\nu = \int_{S_0} T_{\mu\nu}\xi^\mu\xi^\nu \tag{10}$$

with l^μ the future directed normal to S_2. The importance of this form, as we shall see shortly, is that we have relate the solution on some later time slices to their initial data on S_0 Furthermore, the second term is non-negative if we assume $-T_{\mu\nu}v^\mu$ is null or timelike for some future directed timelike v^μ.

From here, we derive the crucial inequality by plugging in the stress energy tensor explicitly

$$\int_{S_1} [(\partial\phi/\partial t)^2 + |\boldsymbol{\nabla}\phi|^2 + m^2\phi^2] \leq \int_{S_0} (\partial\phi/\partial t)^2 + |\boldsymbol{\nabla}\phi|^2 + m^2\phi^2 \tag{11}$$

This ineuqlity will allow us to hit all three targets in our central guideline. It can show uniqueness of solution and the causal propagation of the field, without needing analyticity like Cauchy-Kowalewski theorem requires. It can also show that the solution depends continuously on initial data. However, for this last one, we need a more useful form of (11).

First, the way to show uniqueness is quite similar to how we deal with uniqueness in electromagnetism given boundary conditions. Suppose we have two different solutions ϕ_1 and ϕ_2 with the same initial data, then their difference $\psi = \phi_1 - \phi_2$ vanishes on the initial surface Σ_0 so RHS of (11) vanishes for ψ. Since ϕ is real, LHS is strictly non-negative, so LHS vanishes for ψ at any given time $t_1 > t_0$ for regions within $D^+(S_0)$. The case can be similarly argued for when $t_1 < t_0$, corresponding to $D^-(S_0)$. So we are promised to a unique solution given some fixed initial data.

In addition, it should be obvious from the above argument that variation on the initial data only affects $D(S_0)$, so we hit the causal requirement as well.

Now to hit the requirement that solution depends continously on initial data, we shall massage (11) a bit more. From our KG equation, we observe that all partial derivatives of ϕ with respect to coordinates are also solutions to the KG equation. So with KG equation, we can infact write all higher derivatives of ϕ with just ϕ and $\partial\phi/\partial t$. Then by defining the following norm

$$||\phi||^2_{S_1,k} = \int_{S_1} \{|\phi|^2 + ... + \sum_i \left|\partial^{k_i}\phi\right|^2\} \tag{12}$$

$$|||\phi|||^2_{S_0,k} = \int_{S_0} \{|\phi|^2 + ... + \sum_i \left|D^{k_i}\phi\right|^2\} \tag{13}$$

where the ∂^{k_i} denotes the k_ith spacetime derivatives while D^{k_i} denotes the k_ith space derivatives, we have the inequality that bounds the data on $\S_1$ by initial data on S_0

$$||\phi||_{S_1,k} \leq C_{1,k}|||\phi|||_{S_0,k} + C_{2,k}|||\partial\phi/\partial t|||_{S_0,k-1} \tag{14}$$

where $C_{1,k}$ and $C_{2,k}$ are some constants, and note the $k-1$ index in the last term comes from the fact that we have a derivative within the norm already. Now consider adding data on all S_1 surfaces. Data on each possible S_1 is bounded by the above inequality, so we can integrate over all possible S_1 surfaces where t_1 ranges from t_0 to the maximum value allowed. We shall end up with a gigantic inequality, generalized from (14)

$$||\phi||_{D^+(S_0),k} \leq C'_{1,k}|||\phi|||_{S_0,k} + C'_{2,k}|||\partial\phi/\partial t|||_{S_0,k-1} \tag{15}$$

Then we shall apply the following result. For a subregion A that belongs to a cone of fixed heigh and deficit angle (which is also isometric to a subregion of Euclidean $\mathbb{R}^n$), with $k > n/2$, there exists some constant C such that the numerical value of a smooth function f in A is bounded by the $|| \quad ||_{A,k}$ norm as defined above.

$$\sup_{x\in A} |f(x)| \leq C||f||_{A,k} \tag{16}$$

Make the change $A \to D^+(S_0)$ (or similarly for $D^-(S_0)$) and take $k=3$, we can write down the bound for ϕ and $\partial^m \phi$ in $D^+(S_0)$, which determines all the data in $D^+(S_0)$.

$$\sup_{x\in D^+(S_0)} |\phi| \leq C''_1|||\phi|||_{S_0,k=3} + C''_2|||\partial\phi/\partial t|||_{S_0,k-1=2} \tag{17}$$

$$\sup_{x\in D^+(S_0)} |\partial^m\phi| \leq C''_{1,m}|||\phi|||_{S_0,3+m} + C''_{2,m}|||\partial\phi/\partial t|||_{S_0,2+m} \tag{18}$$

Now, we have shown that KG solutions are also continuously dependent on the initial data, we have hit all three criteria in our guideline (uniqueness, continuous dependence on initial data, and causal propagation), so we may say that KG indeed has a *well posed* IVF.

1.4 Constraints

Before we head on to discussing IVF for GR, recall our first definition of whether a theory possess an IVF. We have dicussed most of the points in the definition, except for the constraints, which become important with gravity.

To discuss it, we shall take a quick detour to electromagnetism, where we have seen constraints right in front of our eyes. Consider the undergraduate Maxwell's equations

$$\nabla \cdot \vec{E} = 4\pi\rho \tag{19}$$

$$\nabla \cdot \vec{B} = 0 \tag{20}$$

$$\partial_t \vec{B} = -\nabla \times \vec{E} \tag{21}$$

$$\partial_t \vec{E} = -4\pi\vec{J} + \nabla \times \vec{B} \tag{22}$$

Of the 4 equations, 2 of them don't really evolve the fields. They are the constraints. In order for the system to actually makes sense, data of the fields at any instant should obey the constraint. This also shows that not all points in the parameter space are accessible. In fact, sometimes, in order to make sure that, after evolving the fields, the parameters will not evolve outside the accessible region, we sometimes have *secondary constraints*, which we shall discuss for gravity in the second part of this paper.

Perhaps in some occasion, we will need even higher constraints to restrict the parameter space further, and perhaps we might eventually realize that no points are accessible in the parameter space for our system. But most of the time, secondary constraint is enough for gauge theory.

Now, back to Einstein equations. Let us first write down the twice contracted Bianchi identity

$$\nabla_\mu G^{\mu\nu} = 0 \tag{23}$$

where $G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R$ is the Einstein tensor. Let us expand this equation a bit

$$\partial_0 G^{0\nu} + \partial_i G^{i\nu} + \Gamma^{\cdot}_{\cdot\cdot} G^{\cdot\cdot} + \Gamma^{\cdot}_{\cdot\cdot} G^{\cdot\cdot} = 0 \tag{24}$$

where we ignore indices for the last two terms because they distract us. Note that in the the last 3 terms, we have at most second order time derivatives, while in the first term, we have 1 time derivative explicitly written out already. So for this equation to hold for any G, there should be at most first time derivative in $G^{0\nu}$, that is, 4 of the 10 Einstein equations are constraints, that they don't evolve the fields. We can think of them as "gauge" freedom.

1.5 Generalization to GR

Since we have this gauge freedom, we need to get rid of these unphysical degrees of freedom by choosing some gauge. Before we choose the gauge, let us sketch the idea of what form we would expect the Einstein equations to be in to possess a *well posed* IVF.

Hopping from KG, we can readily write it in a more general form

$$g^{\mu\nu} \nabla_\mu \nabla_\nu \phi + A^a \nabla_\mu \phi + B\phi + C = 0 \tag{25}$$

where $g^{\mu\nu}$ is some smooth Lorentzian metric such that the metric manifold (M, g) is globally hyperbolic, A^μ is some smooth vector field, and B, C are some smooth function. This can then be generalized to a system of ϕ

$$g^{\mu\nu} \nabla_\mu \nabla_\nu \phi_i + \sum_j (A^a_{ij} \nabla_\mu \phi_j + B_{ij} \phi_j) + C_i = 0 \tag{26}$$

Given arbitrary smooth initial data $(\phi_i, n^\mu \nabla_\mu \phi_i)$ on the initial surface, the above equation has unique solution and a *well posed* IVF. Since gravity is

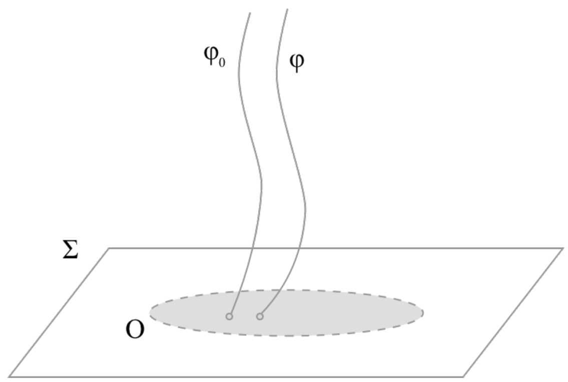

Figure 2: For some small variation of initial data, we can have new unique solution in an open neighborhood $\mathcal{O}$

nonlinear, we would like to throw some nonlinearity in to our theorem. The above result can then be generalized to a quasilinear form, meaning linear in the highest order derivative term,

$$g^{\mu\nu}(x;\phi_j;\nabla_\alpha\phi_j)\nabla_\mu\nabla_\nu\phi_i = F_i(x;\phi_j;\nabla_\alpha\phi_j) \tag{27}$$

Observe in this form, we are allowing our metric to depend on solutions, and we swept the nonlinearity into $\hat{F}_i$.

Theorem 1.4. *Let $(\phi_0)_i$ be any solution to above equation on $\{M,(g_0)^{\mu\nu}[x;(\phi_0)_j;\nabla_\alpha(\phi_0)_j]\}$. Let Σ be a smooth spacelike Cauchy surface in $(M,(g_0)^{\mu\nu})$, then the IVF of above equation is well posed on Σ in the following sense:*
For initial data on Σ sufficiently close to initial data for $(\phi_0)_j$, there exists an open neighborhood $\mathcal{O}$ on Σ such that the above equation has a unique solution ϕ_j in O.

Before we head on, let us now engage our previous discussion on constraints, or getting rid of gauge freedom. Let us choose to work in the hamonic coordinates

$$H^\nu = \nabla_\mu\nabla^\mu x^\nu = 0 \tag{28}$$

Here is how we shall construct this coordinate: first we will choose some initial coordinate $(y^\mu, \partial y^\mu/\partial t)$ on the initial surface Σ, feed it to (28). Observe that (28) has exactly the form as in a generalized linear KG equation for a set of equations (26), so (28) also has a *well posed* IVF on Σ, allowing us to construct unique coordinate $\{x^\mu\}$ from some initial coordinate we choose for Σ.

Then to write (28) out explicitly

$$\begin{aligned} H^\mu &= 0 & (29)\\ \frac{1}{\sqrt{-g}}\partial_\alpha(\sqrt{-g}g^{\alpha\beta}\partial_\beta x^\mu) &= 0 & (30)\\ \partial_\alpha g^{\alpha\mu} + \frac{1}{2}g^{\alpha\mu}g^{\rho\sigma}\partial_\alpha g_{\rho\sigma} &= 0 & (31)\end{aligned}$$

For vacuum Einstein equations in the harmonic coordiantes, we have

$$\begin{aligned} R^H_{\mu\nu} &= 0 & (32)\\ R_{\mu\nu} + g_{\alpha(\mu}\partial_{\nu)}H^\alpha &= 0 & (33)\\ -\frac{1}{2}g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu} + \hat{F}_{\mu\nu}(g,\partial g) &= 0 & (34)\end{aligned}$$

where we swept the nonlinearity into $\hat{F}_{\mu\nu}$. Now all these come together, and the above equation has the form required by (27). So if we have some initial data sufficiently close to vacuum, we can have a *well posed* IVF.

To push this further, we can look at how this local neighborhood $\mathcal{O}$ can be patch together for a global view, and perhaps we can take a look at solutions that are "not very close" to vacuum. However, we will not disucss those in this paper, and shall switch gear to do some calculation with the IVF for gravity instead.

2 (1+3) Split of General Relativity

The purpose of our (1+3) split of gravity is primarily to formulate and solve the IVP. Some further applications and benefits of this split is that we can integrate the Einstein equatoins by numerical codes, allowing us to probe more systems.

In this part, we will also see that we can characterize degrees of freedom better with the split, and we can characterize isolated system and their associated symmetry groups and conserved quantities, like energy/mass, momentum (linear and angular).

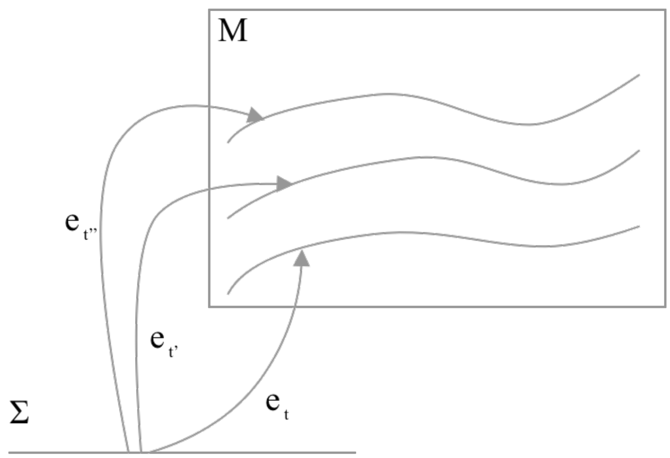

Figure 3: Imagine the spacetime (M, g) being foliated into sheets, each representing a constant time slice with (Σ, h)

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Our first goal is to write the Einstein equations in the form of a constraint Hamiltonian system. To do it, we will start by decomposing the metric, and we will lead up to the decomposition of curvature, and finally we will arrive at the Hamiltonian system, and we will end at introducing the constraint algebra.

2.1 Decomposition of Metric

To start with, let us assume the spacetime to be topologically a product: $M \approxeq \mathbb{R} \times \Sigma$, which would be the same to say that this is globally hyperbolic, having Cauchy surface, is stably causal, and can assign time function, all good properties physicists like.

The idea here is that we want to foliate the spacetime and view it as history of spaces (Σ, h), where h is the induced metric on Σ (see Figure 3). Also we shall note that h is Riemannian.

Mathematically, this means that we have a 1-parameter family of embedding

$$e_t : \quad \Sigma \quad \longrightarrow M, \qquad t \in \mathbb{R} \tag{35}$$

$$\Sigma_t \quad \equiv e_t(\Sigma) \subset M \tag{36}$$

This naturally induces a split of the tangent spaces T_pM into parallel and normal to Σ components

$$\begin{aligned}
P_{\parallel}: \quad TM &\longrightarrow T_{\parallel}M && (37)\\
X &\longmapsto X + ng(n, X) && (38)\\
P_{\perp}: \quad TM &\longrightarrow T_{\perp}M && (39)\\
X &\longmapsto -ng(n, X) && (40)
\end{aligned}$$

where n is the timelike normal to Σ. It should be also noted that this can be easily extended to the dual tangent spaces T_pM^* and to the various products of T_pM and T_pM^*.

With this mapping, we can write the induced metric as

$$h = P_{\parallel}g = g + \underline{n} \otimes \underline{n} \tag{41}$$

where $\underline{n} \equiv g(n, \cdot)$ is the musical flat operator.

We may then split the covariant derivative in a similar fashion. Let $X, Y \in \Gamma T_{\parallel}M$ be vector fields in the section of bundle $T_{\parallel}M$, we have

$$\nabla_X Y = P_{\parallel}(\nabla_X Y) + P_{\perp}(\nabla Y) = D_X Y + nK(X, Y) \tag{42}$$

Claim 2.1.

1. *$K \in \Gamma(T^*_{\parallel}M \otimes T^*_{\parallel}M)$ is a symmetric vector field;*

2. *D is the Levi-Civita connection on $T_{p\parallel}M$ w.r.t. the induced metric h.*

Proof.

1. Here we need to show that $K(fX, Y) = K(X, fY) = fK(X, Y)$. This can be shown by definition

$$\begin{aligned}
K(X, Y) &= -g(n, P_{\perp}\nabla_X Y) && (43)\\
&= -g(n, \nabla_X Y) && (44)\\
&= -g(n, \nabla_Y X + [X, Y]) && (45)\\
&= -g(n, \nabla_Y X) && (46)
\end{aligned}$$

where in the last step we assumed the spacetime we are dealing with is torsion free.

2. For this one, we need to show that D is a connection. We can do so by checking the definition of connection.
   First, it is metric compatible

$$D_X h = P_\parallel \nabla_X (g + \underline{n} \otimes \underline{n}) = 0 \tag{47}$$

   This can be shown through $\nabla_X g = 0$, and that the Leibniz rule pulls out a $\underline{n}$ from the second term to be annhilate by the parallel projection.

□

In this decomposition, $K(X, Y)$ is called the extrinsic curvature of the hypersurfaces Σ in M. As recalled in class, it has a relation to the Lie derivative via

$$\mathcal{L}_n h = 2K \tag{48}$$

Next, since we need some concept of how time flows in this (1+3) split, let us decompose $\partial/\partial t$. This can be done by introducing two parameters, the lapse function $\alpha \in C^\infty(M)$, and the shift vector field $\beta \in \Gamma T_\parallel M$. A visualization of the two parameter is in Figure 4

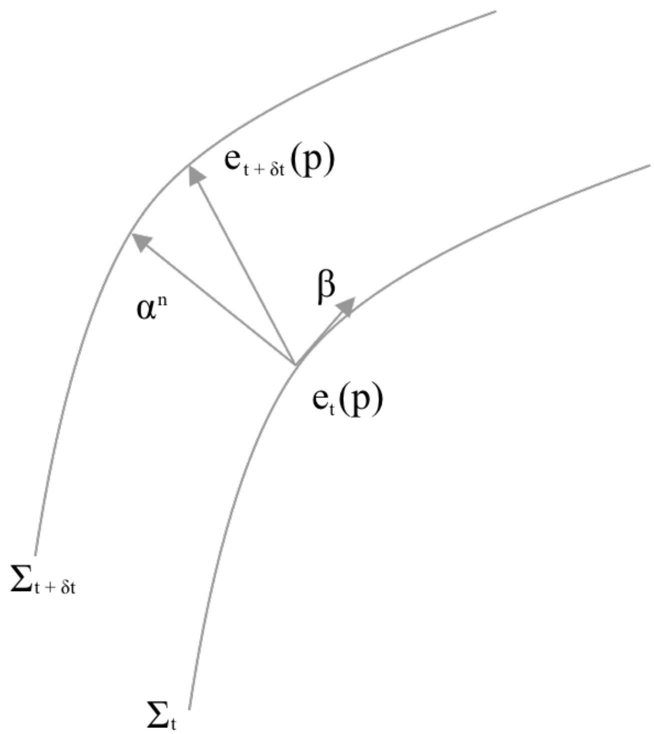

Figure 4: Visualization of α and β

With α and β, we will be able to write down the transformation between coordinate $\partial/\partial x^\mu$ and an adpated basis where we require that $e_0 \equiv n$ and orthonormality $g(e_a, e_b) = \delta_{ab}$. And let the dual basis w.r.t $\{e_\nu\}$ be $\{\theta^\mu\}$ such that $\theta^\nu(e_\mu) = \delta^\nu_\mu$. We will skip the details here to simply write down the decomposed metric

$$
\begin{aligned}
g &= -\theta^0 \otimes \theta^0 + \sum_1^3 \theta^a \otimes \theta^a && (49)\\
&= -(\alpha^2 - h(\beta,\beta))\mathrm{d}x^0 \otimes \mathrm{d}x^0 && (50)\\
&\quad +\beta_m(\mathrm{d}x^0 \otimes \mathrm{d}x^m + \mathrm{d}x^m \otimes \mathrm{d}x^0) && (51)\\
&\quad +h_{mn}\mathrm{d}x^m \otimes \mathrm{d}x^n && (52)
\end{aligned}
$$

We can observe this decomposition closely to see that the first and second term consists of 4 components, while the third term consists of 6 components. And we should immediately recall that Einstein equations have 6 time evolution and 4 constraints, meaning that h_{mn} will be determined by time evolution, while β_m and α act like gauge functions. They describe roughly how fast the constant time slices will move in normal and tangential directions. An analogy to them is the gauge fields in electromagnetism.

2.2 Decomposition of Curvature

The idea for decomposing the curvature is essentially replacing ∇ with D. This can be done by substituting $\nabla_X Y = D_X Y + nK(X,Y)$ into the Riemann tensor. To simply skip ahead, we have

$$Riem^{\nabla}(\omega, Z, X, Y) = Riem^{D}(\omega, Z, X, Y) + K(\omega, X)K(Z, Y) - K(\omega, Y)K(Z, X) \tag{53}$$

To write it in the component form for $d = 2$, we have

$$Riem^{\nabla}_{abcd} = Riem^{D}_{abcd} + K_{ac}K_{bd} - K_{ad}K_{bc} \tag{54}$$

This is known as Gauss's *theorema egregium*, which sounds so cool but I haven't had a chance for any direct application of it.

From here we can obtain the Ricci scalar

$$R^{\nabla} = R^{D} + (K_{ab}K^{ab} - (K^{c}_{c})^2) + 2\nabla \cdot v \tag{55}$$

where $v = nK^{c}_{c} - a$ but isn't going to be much relevant since the term involving it is a surface term. Using the DeWitt metric

$$G^{abcd} = \frac{1}{2}(h^{ac}h^{bd} + h^{ad}h^{bc} - 2h^{ab}h^{cd}) \tag{56}$$

Then we can write the Ricci scalar as

$$R^{\nabla} = G(K, K) + R^{D} + (surface) \tag{57}$$

where $G(K, K)$ is the "kinetic" term and the R^D term "potential". We now have all the ingredients needed. The Einstein-Hilbert action can now be written as

$$L_{EH}(h, \dot{h}, \alpha, \beta) = G(K, K) + R^{D} \propto \sqrt{h}\mathrm{d}^4 x \tag{58}$$

Before proceeding to next section, let us write down a recipe for calculating with IVF of GR in vacuum:

1. Choose 3-manifold Σ;

2. Pick a pair of (h, K) such that the constraints are satisfied;

3. Pick any 4 non-zero functions α, β^m to make Σ move forward;

4. Evolve (h, K) according to $\dot{h} = \mathcal{L}_{\partial/\partial t} h$ and $\dot{K} = \mathcal{L}_{\partial/\partial t} K$

2.3 Hamiltonian Formulation and Constraints

Since now we have the Lagrangian in the split form, we can write down the canonical momenta associated with the system

$$\Pi_\alpha = \frac{\partial L_{EH}}{\partial \dot{\alpha}} = 0 \tag{59}$$

$$\Pi_{\beta^n} = \frac{\partial L_{EH}}{\partial \dot{\beta}^n} = 0 \tag{60}$$

$$\tag{61}$$

These are the so called *primary constraints*. Having the canonical momenta equal to 0 does not mean that the system cannot move. These correspond to gauge symmetry, and the above constraints just mean that moving in the these corresponding directions in the phase space does not cost any action. They are not physical degrees of freedom.

Furthermore, by having the *primary constraints*, we are effectively revoming (α, Π_α) and (β^n, Π_{β^n}) from the canonical degrees of freedom. Perhaps another good way to consider α and β^n is to think about them as Lagrange multipliers.

To obtain Hamiltonian, we need one more thing, expressing $\dot{h}$ in terms of $\Pi^{ab} = \frac{\partial L_{EH}}{\partial \dot{h}_{ab}} = G^{abcd} K_{cd}$

$$\dot{h} = \mathcal{L}_{\partial/\partial t} h \tag{62}$$

We can use the fact that $\partial/\partial t = \alpha n + \beta$ to write $n = \frac{1}{\alpha}(\frac{\partial}{\partial t} - \beta)$, and then use $K = \frac{1}{2}\mathcal{L}_n h$ to get that

$$\dot{h}_{ab} = (\mathcal{L}_\beta)_{ab} + 2\alpha K_{ab} \tag{63}$$

$$= D_a \beta_b + D_b \beta_a + 4\alpha\sqrt{h} G_{abcd} \Pi^{cd} \tag{64}$$

Recall from the part one that we briefly introduced the concept of *secondary constraint*. Here, in order for our *primary constraint* to be preserved over time, we need to enforce that

$$\dot{\Pi}_\alpha = \{\Pi_\alpha, H_0\} = -\frac{\delta H_0}{\delta \alpha} = 0 \tag{65}$$

$$\dot{\Pi}_{\beta^n} = \{\Pi_{\beta^n}, H_0\} = -\frac{\delta H_0}{\delta \beta^n} = 0 \tag{66}$$

$$\tag{67}$$

These two are the *secondary constraints*. Altenativaly, we can write them as

$$2\hat{G}^{-1}(\Pi,\Pi) - \frac{1}{2}R^D\sqrt{h} \quad = \quad 0 \tag{68}$$

$$-2D_a\Pi^{ab} \quad = \quad 0 \tag{69}$$

where $\hat{G}^{-1}$ is the inverse DeWitt metric multiplied by $\sqrt{h}$. The first equation is also known as Hamiltonian or scalar constraint, which the second one vecot or diffeomorphism constrain.

As promised before, there will be no further constraint. We can check this by taking further constraint and see if we will evolve the system out of accessible phase space. Again, we need to note that having only up to secondary constraint is not a generic fact for all gauge theory.

2.4 Constraint Algebra

Finally, let use remark that the Hamiltonian is a functional of h, Π, α, β, being split into the scalar and vector portion. We can write this in a general form

$$H[h,\Pi,\alpha,\beta] = C_s(\alpha)[h,\Pi] + C_v(\beta)[h,\Pi] + surface \tag{70}$$

where C_s stands for scalar constraint and C_v stands for vector constraint. They have expressions

$$C_s(\alpha)[h,\Pi] \quad = \quad \int_\Sigma \mathrm{d}^3x\alpha[2\hat{G}^{-1}(\Pi,\Pi) - \frac{1}{2}R^D\sqrt{h}] \tag{71}$$

$$C_v(\beta)[h,\Pi] \quad = \quad \int_\Sigma \mathrm{d}^3x\beta^a[-2h_{ab}D_c\Pi^{bc}] \tag{72}$$

Note that we have gotten rid of the surface terms. Of course it is legal to keep track of them from the beginning and carry them around for each step. However, this is in general messy and unnecessary. We can ignore them in the action. At the end, we can reconsider them in the Hamiltonian. By then we just need to add corresponding surface terms that will make the Hamiltonian functionally differentiable w.r.t. h and Π. We note that actions should give decent equation of motions anyways. If the Hamiltonian turns out nasty due to boundary, we will just go back and massage it better.

Let us consider a simple example when $\alpha = 0$, that is, we stay on a constant time slice. Use the Poisson bracket relation we have above

$$\begin{aligned} \dot{h}_{ab} &= \{h_{ab}, C_v(\beta)[h,\Pi]\} && (73)\\ &= \frac{\delta}{\delta\Pi^{ab}}\int_\Sigma \mathrm{d}^3x\beta^a[-2h_{ab}D_c\Pi^{bc}] && (74)\\ &= D_a\beta_b + D_b\beta_a && (75)\\ &= \mathcal{L}_\beta h && (76)\end{aligned}$$

Perhaps not so surprisingly, when we have a constant time slice, the Hamiltonian will generate the Lie derivative on the slice. And again we note that if surface term is hanging around, we need to add corresponding terms to cancel them out.

And finally, we introduce the constraint algebra, which is universal and is very important for further exploration in canonical quantum gravity, which we will not have the time to delve into in this paper.

$$\begin{aligned} \{C_v(\beta), C_v(\beta')\} &= C_v([\beta,\beta']) && (77)\\ \{C_v(\beta), C_s(\alpha)\} &= C_s(\beta(\alpha)) && (78)\\ \{C_s(\alpha\, C_s(\alpha')rbrace &= \epsilon C_v(\alpha(\mathrm{d}\alpha')^\sharp - \alpha'(\mathrm{d}\alpha)^\sharp) && (79)\end{aligned}$$

where in the last line, $\epsilon = +1$ for Riemannian, and $\epsilon = -1$ for Lorentzian, and $\mathrm{d}\alpha)^\sharp)$ is the musical sharp $(\mathrm{d}\alpha)^\sharp = g^{ab}\frac{\partial\alpha}{\partial x^b}\frac{\partial}{\partial x^a}$

3 Conclusion

Overall the canonical formalism is a very fundamental topic. In this paper we scrapped the surface and tried to see how it is built. We started with a more rigorous understanding of the *well posed* IVF, and developed our understanding through simple theory such as KG, and then we generalize step by step, introducing nonlinearity into our build up to incorporate gravity.

However, we did not show exactly how IVF is patched together for gravity beyond a simple local neighborhood $\mathcal{O}$, which should be an interesting topic to further explore.

In the second part of the paper, we delve into actually decomposing gravity into a (1+3) split, starting with metric, then covariant derivative, and fianlly curvature and Einstein-Hilbert action. We discuss the concept of constraints, and how it is related to the degrees of freedom in canonical formulation of gravity, and eventually we end on a very curious form of constraint algebra, which we did not develop further than simply stating it.

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