Ryu-Takayanagi Formula

Nadie Yiluo LiTenn
University of California, Santa Barbara, California 93107
(Dated: December 12, 2021)

In recent developments, we see the rapid connections between condensed matter physics and gravity theory that reveals deep physics. One example of this is the Ryu-Takayanagi formula, which describes the entanglement entropy for a subregion on the boundary of a bulk-boundary theory, in terms of the geometry in the bulk. In this review, we shall go over a short review of the entanglement entropy, discuss the proposal and restrictions of the Ryu-Takyanagi formula, and describe a Euclidean path integral derivation for the Ryu-Takayanagi proposal.

I. ENTANGLEMENT ENTROPY

We start with a small review on the entanglement entropy, otherwise known as the the von Nuemann entropy or the fine grained entropy. In a finite quantum mechanical system, whose density matrix is $\rho = |\psi\rangle \langle\psi|$, we can define the von Neumann entropy

$$S_{vN} = -\operatorname{tr}(\rho \log \rho) \tag{1}$$

This notion has an origin from the classical Shannon entropy, which is essentially a special case that takes only the classical portion of a density matrix. However, as we have a larger quantum system, such as an entire quantum field, the above quantity becomes hard to calculate due to the log inside the trace. However, this difficulty can be avoided by introducing the Renyi entropy

$$S_n = -\tfrac{1}{n-1} \log \operatorname{tr}(\rho^n) \tag{2}$$

where the n label means the n-th Renyi entropy. We still have a log but now it is outside of the trace, so we are expecting this to be a better treatment. However, we should note that the density matrix in quantum field theory is not an exactly well defined quantity, but it is largely believed to behave nicely and provide sensible results, so we shall charge forward.

In the above definition, Renyi entropy takes discrete values n, we can analytically continue it to all positive real numbers, and take the limit of $n \to 1$

$$\begin{aligned}
\lim_{n\to 1} S_n &= -\lim_{n\to 1} \frac{\mathrm{d}}{\mathrm{d}n} \log \operatorname{tr}(\rho^n) && (3)\\
&= -\lim_{n\to 1} \frac{\frac{\mathrm{d}}{\mathrm{d}n} \operatorname{tr}\rho^n}{(\operatorname{tr}\rho)^n} && (4)\\
&= -\lim_{n\to 1} \frac{\mathrm{d}}{\mathrm{d}n} \operatorname{tr}\big(e^{n\log\rho}\big) && (5)\\
&= -\operatorname{tr}(\rho\log\rho) = S_{vN} && (6)
\end{aligned}$$

as desired. From second to third line, we use the fact that $\operatorname{tr}\rho = 1$ for a porperly normalized density matrix.

II. RYU-TAKAYANAGI

In this section we will describe the holographic entanglement entropy proposal by Ryu and Takayanagi[7]. Long before Ryu and Takayangi, it was already pointed out that the entanglement entropy resembles the black hole entropy[8] [1]. When we consider a quantum many-body system AB and restrict ourself to a subsystem A, we will find the entanglement entropy of the subsystem to be

$$S_A = \kappa M^2 A \tag{7}$$

where κ is a constant dependent on the specific system that we study, M correspond to the UV cutoff (e.g. for a lattice with spacing a, $M \sim \frac{1}{a}$), and A is the area of the boundary between subsystem A and B, which may or may not be disjoint.

We recognize that this shares the same area law as the black hole entropy

$$S_{bh} = \frac{1}{4} M_{pl}^2 A \tag{8}$$

where M_{pl} is the Planck mass. So it is as if we have some kind of horizon-like screen in the quantum many-body system. But we also note that this screen is put in by hand arbitrarily. So the question of which surface it should be is also one of our concern, especially when there is ambiguity in the choices.

In 2006, Ryu and Takayanagi proposed a solution to this question. The idea is that suppose we have a bulk-boundary theory, where on the boundary we have quantum fields without gravity, and in the bulk we have quantum gravity theory. Let us devide the spatial boundary AB into subregion A and B, and calculate the entanglement entropy of A. It is then proposed to be

$$S_A = \frac{\text{Area}[\gamma_A]}{4G_N} \tag{9}$$

where γ_A is the minimal surface in the bulk that is homologous to the subregion A on the boundary, and G_N is the Newton's constant. In the simple case of a 2+1D bulk, the minimal surface is just a geodesic.

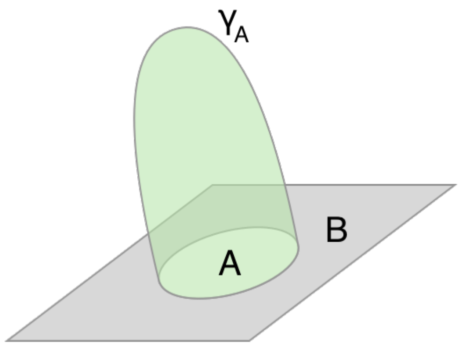


FIG. 1: The rectangular surface is the boundary divided into subregion A and B, and γ_A is the minimal surface in the bulk, bounded by the same boundary of subregion A, $\partial\gamma_A = \partial A$

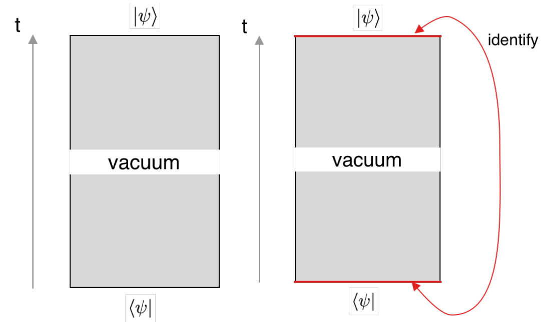


FIG. 2: Left: representation of density matrix in terms of path integral, where we time evolve from vacuum to obtain our bra and ket states. Right: by identifying the two red sides, we obtain $\operatorname{tr}\rho$, which is defined to be 1 for any normalized density matrix.

A. Restriction of Ryu-Takayanagi

Ryu-Takayanagi formula is restricted in the following sense[4]: it requires the bulk theory to be classical Einstein theory and bulk spacetime to possess a time reflection symmetry under which the boundary spatial region A is invariant.

There are three ways we can build our improvements on Ryu-Takayangi based on this understanding of its restriction, which we shall not cover in details in this short review.

First, we can move from classical theory. Then it is necessary that we include quantum correction terms[3]. This will be given by the similar Euclidean path integral derivation as the one we shall review in this paper.

If we move away from the Einstein theory, which we consider to be an effective theory, then we will need to include higher derivative terms in the action.

$$S_{bulk} = \frac{1}{16\pi G_N^{(d+1)}} \int \sqrt{g}(R + \frac{d(d-1)}{l_{AdS}^2} \tag{10}$$

$$+ \sum \alpha_k \nabla^{2k} R + \mathcal{L}_{natter}) \tag{11}$$

In doing this, we are getting more "stringy", where the string coupling term starts to get significant.

Finally, if we leave out the time reflection symmetry, we will go into, instead of a static geometry, a time dependent one. This is in fact known as the Hubeny-Rangamani-Takayanagi formula[2, 5], where a Lorentzian approach is used, which requires no time reflection symmetry.

III. DERIVATION OF THE RYU-TAKAYANAGI FORMULA

The path integral derivation was given by Lewkowycz and Maldacena[6]. Though the full details require the framework of AdS/CFT, we shall content ourselves with a model of gravity in the box for our derivation. So our setup is to have finite $S^2 \times \mathbb{R}$ for our boundary condition, where $\mathbb{R}$ is the time direction. We will consider the subregion $A \subset S^2$

First, we know that we can obtain any arbitrary quantum state $|\psi\rangle$ by time evolving a vacuum state $|\Omega\rangle$ that is in the far past. And by time reflection symmetry, we can similarly obtain $\langle\psi|$ by time evolving backwards on a vacuum in the far future. With this construction, we can represent our desnity matrix as in Figure 2. We note that everything we manipulate with right now lives on the boundary. If we are to obtain the quantity $\operatorname{tr}\rho$, we then simply identify the top and bottom sides. Therefore, in some sense, each top and bottom side represents an index of the density matrix. If there are no free top and bottom sides in the diagram (i.e. all of them are identified with another side), then we are expecting a number instead of a matrix.

Our next step is to obtain the density matrix for the subregion A that we are considering. To do this, we will trace out the degrees of freedom of subregion B from the full density matrix. That is, we will do a partial tracing to get the reduced density matrix for A.

$$\rho_A = \operatorname{tr}_B \rho = \sum_i \langle\psi_{B,i}|\,\rho\,|\psi_{B,i}\rangle \tag{12}$$

This can be similarly represented with a diagram (Figure 3). The idea is that we identify the a subregion in the top and bottom sides.

From here, it is straightforward to obtain the $\operatorname{tr}(\rho_A^2)$ (see Figure 4) and similarly for $\operatorname{tr}(\rho_A^n)$, which we shall not draw. We will then notice that, if we follow the purple trajectory in Figure 4 there is a singularity correspond to the conical surplus. So we learn from this that the boundary contains a conical singularity.

Let us pause here and remind ourselves that our ultimate goal is to calculate the entanglement entropy, which requires analytic continuation of integer n in the Renyi entropy to all positive real numbers. Now, it is not exactly clear what this analytic continuation means with the pictures we are working with, so we shall switch to a bulk picture instead.

We see from the above manipulation that the boundary exhibits a replica symmetry. For example, for $n = 2$ case, we have a $\mathbb{Z}_2$ symmetry, which can be explicitly seen through Figure 5. The key is that we now have a fixed point under the replica symmetry, and this point carries the conical singularity.

To proceed, we will make an assumption that the bulk saddles will carry the same kind of replica symmetry. So instead of working in the full bulk manifold $\mathcal{M}$, we can work in a quotient manifold $\tilde{\mathcal{M}}_n/\mathbb{Z}_n$, and we identify all n poitns under the replica symmetry. Now the boundary is smooth, and we have essentially moved the singularity into the bulk quotient space $\tilde{\mathcal{M}}_n$

Now let I_n be the Euclidean action for full, smooth n-replica in the bulk. Under saddle point approximation, we can write

$$\mathrm{tr}(\rho^n) \approx e^{-I_n} \tag{13}$$

where we drop the subscript A as we will exclusively be talking about the subregion.

For simplicity, we define

$$\hat{\rho} \equiv \frac{\rho}{\mathrm{tr}\,\rho} \tag{14}$$

so we can write

$$\mathrm{tr}\,\hat{\rho}^n = \frac{\mathrm{tr}\,\rho^n}{(\mathrm{tr}\,\rho)^n} = e^{-(I_n - nI_1)} \tag{15}$$

We can also define a quotient action, which integrates over the quotient space $\tilde{\mathcal{M}}_n$.

$$\tilde{I}_n \equiv \frac{I_n}{n} \tag{16}$$

We should however note that $\tilde{I}_n$ is not the Einstein-Hilbert action for $\tilde{M}_n$ because we have not considered

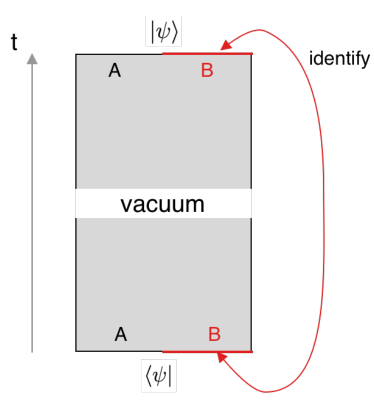


FIG. 3: We obtain the reduced density matrix for subregion A by tracing out the degrees of freedom of B. This also corresponds to possible information that observer in A may obtain when they are ignorant of what is going on in B.

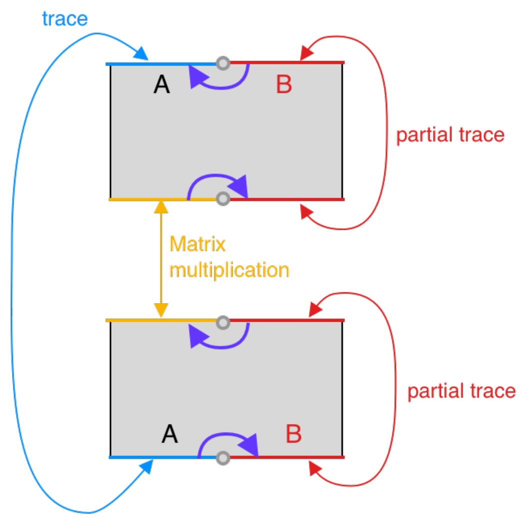


FIG. 4: A diagramic representation of $\mathrm{tr}(\rho_A^2)$, where we have two copies of ρ and "stitch" together subregions accodingly.

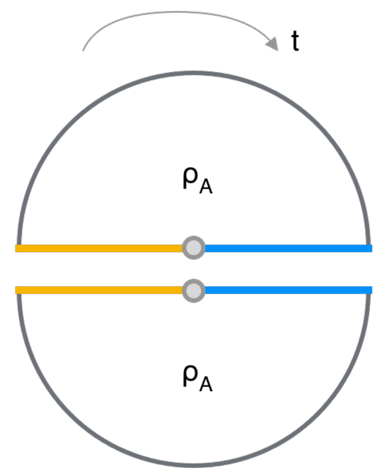


FIG. 5: A $\mathbb{Z}_2$ symmetry is explicit if we represent it this way, and in the center we have a fixed point under the symmetry transformation, and this point is where the conical singularity lies.

contribution from the conical singularity in $\tilde{\mathcal{M}}_n$. With these notations, we can write the n-th Renyi entropy as

$$S_n = -\frac{1}{n-1}\log \hat{\rho}^n \tag{17}$$
$$= \frac{n}{n-1}(\tilde{I}_n - I_1) \tag{18}$$

We can then take the limit of $n \to 1$ to obtain the von

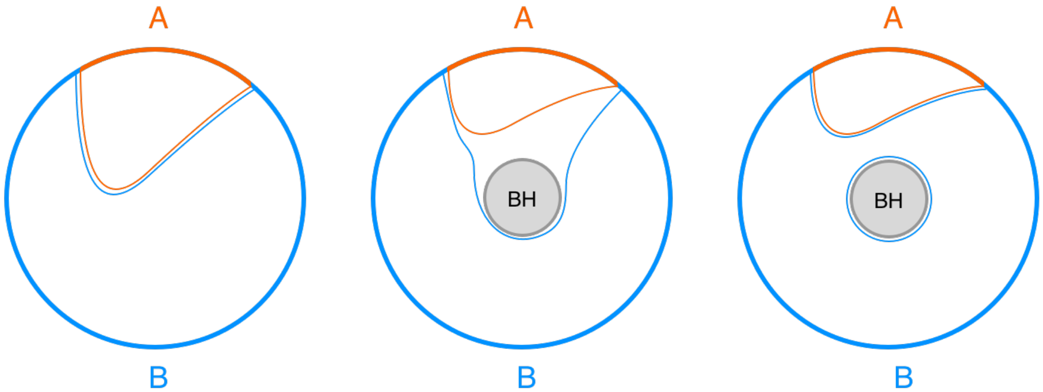


FIG. 6: Left: the minimal surfaces for subregion A and B coincide in the bulk due to absence of black hole. Middle: minimal surface based on the homotopic condition. Right: minimal surface based on the homology constraint.

Neumann entropy.

$$S_{vN} = \lim_{n\to 1} S_n \tag{19}$$
$$= \lim_{n\to 1} \frac{\tilde{I_n} - I_n}{n-1} \tag{20}$$
$$= \frac{\mathrm{d}}{\mathrm{d}n}\tilde{I}_n\Big|_{n=1} \tag{21}$$

We will pause here to think about what $\tilde{I_n}(\tilde{\mathcal{M}_n})$ is exactly. We already know that this quantity is smooth from its definition, but the Einstein-Hilbert action for the quotient manifold includes a term where the Ricci scaler $R(\tilde{\mathcal{M}_n})$ gives a delta-function. So in order to get a smooth $\tilde{I_n}(\tilde{\mathcal{M}_n})$, we need

$$\tilde{I_n}(\tilde{\mathcal{M}_n}) = I_{EH}(\tilde{\mathcal{M}_n}) \tag{22}$$
$$+ \text{ (a term to cancel the singularity)} \tag{23}$$

We recall that in the Einstein-Hilbert action, we have

$$I_{EH} \sim \frac{\sqrt{g}R}{16\pi G} \tag{24}$$

where $\sqrt{g}R \sim 2\alpha\delta(\text{at singularity})$. α here is the conical defect angle. We can then write the singularity in the Einstein-Hilbert action as

$$\sim -\frac{\alpha/2\pi}{4G}\delta(\text{at singularity}) \tag{25}$$

After n-folding, we are left with $\frac{2\pi}{n}$ conical opening, so the conical defect is $\alpha = 2\pi(1-\frac{1}{n})$. Here, we also see the advantage of working with the bulk picture instead of the boundary picture. Analytic continuation is more natural in the bulk, specifically from this equation, where $n \in \mathbb{R}$ just means that we have a continuous spectrum of the conical defect. Put this into above and integrate over the quotient manifold, the delta-function simply becomes the area (of the codimensions that we surpressed in our pictures) at the conical singularity. And we have the quotient action

$$\tilde{I_n}(\tilde{\mathcal{M}_n}) = I_{EH}(\tilde{\mathcal{M}_n}) + \frac{A_{conical}}{4G}(1-\frac{1}{n}) \tag{26}$$

We can then substitute this back into our calculation of the von Neumann entropy to obtain

$$S_{vN} = \frac{\mathrm{d}}{\mathrm{d}n}\tilde{I}_n\Big|_{n=1} \tag{27}$$
$$= \frac{\mathrm{d}}{\mathrm{d}n}I_{EH}(\tilde{M}_n)\Big|_{n=1} + \frac{1}{4Gn^2}A_{conical}\Big|_{n=1} \tag{28}$$
$$+\frac{1-\frac{1}{n}}{4G}\frac{\mathrm{d}}{\mathrm{d}n}A_{conical}\Big|_{n=1} \tag{29}$$

First term vanish because it is smooth and on stationary, as we are working at the saddle. The third term also vanish when we take the $n=1$ limit, so we are left with

$$S_{vN} = \frac{A_{conical}}{4G}\Big|_{n=1} \tag{30}$$

Now we need to ponder what is $A_{conical}$ at $n=1$. As we extremize the action to find the saddle point, we are varying the location of the conical singularity, so what we are finding is in fact the location of a surface that extremizes the von Neumann entropy.

$$S_{vN} = \frac{A(\text{extremal surface})}{4G} \tag{31}$$

And this concludes the path integral derivation of the Ryu-Takayanagi formula.

A. Homology Constraint

We should discuss briefly the homology constraint on choosing the Ryu-Takayanagi surface.

Definition III.1. *We say a surface γ_A is homologous to the subregion A, denoted by $\gamma_A \sim A$ if there exists a bulk region $r \subseteq \mathcal{M}$ such that*

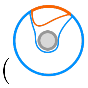
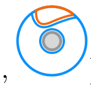
$$\partial r = A \cup \gamma_A \tag{32}$$

In this language, we can also write the Ryu-Takayanagi formula as

$$S(A) = \min_{\gamma_A \sim A} \frac{\text{Area}[\gamma_A]}{4G_N} \tag{33}$$

In other words, as we choose the Ryu-Takayanagi surface, we always have to follow the homology constraint. Note that this is not the same as the requirement that the surface is homotopically equivalenet to the boundary region A, in which case we have a stronger condition that we must be able to smoothly deform the Ryu-Takayanagi surface to the boundary region A. In fact, homotopy condition implies the homology constraint, but not the other way.

Let us consider an example (Figure 6). For the case when there is no black hole in the bulk, the extremal surface γ_A and γ_B conincide in the bulk. When there is a blakc hole, there are two ways we can draw the Ryu-Takayanagi surface as shown in the middle and right panel in Figure 6. The middle panel corresponds to when we have homotopic condition, and the right panel corresponds to when we have just the homology constraint. In the Ryu Takayanagi formula, we choose the minimal configuration.

$S(A) \sim \min($  ,  $)$ (34)

In the BTZ black hole, we have this kind of situation, and as we change the relevant parameter, we will have a phase transition from one configuration to another in calculating our entanglement entropy.

[1] Luca Bombelli, Rabinder K. Koul, Joohan Lee, and Rafael D. Sorkin, *Quantum source of entropy for black holes*, Phys. Rev. D **34** (1986), 373–383.

[2] Xi Dong, Aitor Lewkowycz, and Mukund Rangamani, *Deriving covariant holographic entanglement*, Journal of High Energy Physics **2016** (2016), no. 11.

[3] Thomas Faulkner, Aitor Lewkowycz, and Juan Maldacena, *Quantum corrections to holographic entanglement entropy*, Journal of High Energy Physics **2013** (2013), no. 11.

[4] Matthew Headrick, *Lectures on entanglement entropy in field theory and holography*, 2019.

[5] Veronika E Hubeny, Mukund Rangamani, and Tadashi Takayanagi, *A covariant holographic entanglement entropy proposal*, Journal of High Energy Physics **2007** (2007), no. 07, 062–062.

[6] Aitor Lewkowycz and Juan Maldacena, *Generalized gravitational entropy*, Journal of High Energy Physics **2013** (2013), no. 8.

[7] Shinsei Ryu and Tadashi Takayanagi, *Holographic derivation of entanglement entropy from the anti–de sitter space/conformal field theory correspondence*, Physical Review Letters **96** (2006), no. 18.

[8] Mark Srednicki, *Entropy and area*, Physical Review Letters **71** (1993), no. 5, 666–669.