PHYS 150 Term Paper - Entropy, Information, and the Universe

With a small detour to Maxwell Equations

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1 What is Entropy

Entropy is a confusing concept. When we first learnt thermodynamics, the second law was stamped into our head stating that entropy always increases in an isolated system. However, it doesn't help one internalize the idea of entropy. To correctly approach entropy, we must understand that it comes from the divorce of reversible and irreversible processes.

Let us first consider an arbitrary process in nature. This process will bring all the objects concerned from a deterministic initial state A to a deterministic final state B. With a bit of faith, we conclude that this process is either reversible or irreversible and without a third possibility.

Now, if the system remains isolated, and that after all being done, going back from B to A is not possible, we call this an irreversible process, and henceforth the nature must possess some form of preference of state B over state A. And we infer that nature does not allow the processes for which the final state possesses a smaller preference than the initial state.

Along the same line of thought, the reversible processes must have their final and original states possessing the equal preferences to the nature such that there exists a symmetry between going forward and backward.

One example of the irreversible process is the simple heat conduction from a hotter to a colder object. There is a preference for heat to go in a certain direction. So we wonder, is there a quantity that could measure the nature's preference for every state concerned at the moment. We would like this quantity to remain unchanged when the system considered is undergoing reversible transformation, and increase monotonically when undergoing irreversible transformation.

Well, Clausius made it. He found this quantity, and it was then called entropy. To push it further, we are going to guess an expression for it, and we consider again the simple heat conduction process. We have three relevant parameters in this process: heat Q being transferred, temperature T_1 of the hotter object, and temperature T_2. Since temperature and heat both have the dimension of energy, we speculate that entropy is dimensionless, and guess the form $\frac{T}{Q}$. But oh no. When there is no heat transferred, this term blows up to infinity, which is not what we want, so instead we will guess $\frac{Q}{T}$. And we can write

$$\Delta S = -\frac{Q}{T_1} + \frac{Q}{T_2} \geq 0 \tag{1}$$

where ΔS is the change of entropy in this isolated system. We then remember that, if this process is a reversible process, then $\Delta S = 0$, and if irreversible, then $\Delta S > 0$, and we have the second law that we introduced at the beginning: entropy always increases in an isolated system. Put a negative to the above and call Q the amount of heat lost by an object with temperature T in the system, we have

$$\sum_i \frac{Q_i}{T_i} \leq 0 \tag{2}$$

the inequality of Clausius, and a mathematical way of saying entropy is always increasing. Note that this is by convention. If we start by saying we want a quantity that will decrease, instead of increase, monotonically when the system undergoes irreversible transformation, then entropy will always be decreasing, but the corresponding physics is the same, so we don't care and will go with always increasing. With the expression of entropy relating to heat and energy, and the knowledge that $-pdV$ is the work done by the environment on a system, we can re-express the conservation of energy for thermal physics by writing $dE = TdS - pdV$. Familiar? Sure it is. And we have both laws for thermodynamics now.

Now back to our initial claim that physics is differentiated into reversible and irreversible processes. With entropy tied into the heart of this claim, we are ready to understand why such a differentiation is most essential to physics, for a greater similarity is shared among reversible processes than to any irreversible ones.

Consider the differential equations that describe any reversible processes, partial derivative of time must always enter as an even power to ensure the same physics in a time reversal. Among these processes include the ones that are infinitely slow and all intermediate states are in equilibrium, where time in general plays no role, and the partial derivative of time enters as zeroth order, which is also even. Additionally, as Helmholtz pointed out, all reversible processes can be solved by the principle of least action, not just the classical mechanics we are familiar of. Then for And we arrive at a bold claim: all reversible processes are solved!

Now we have touched on thermal and classical mechanics, but we claimed that reversibility is crucial for all physics, so we will soon see an example in electromagnetism and march into discussion of quantum.

2 Guessing the Maxwell Equations

To study electromagnetism, we can hardly bypass Maxwell, unless we go into quantum field theory. So in the fly by night spirit, let us first guess the Maxwell equations. With some aid from an experimental fact, we can start from the Lorentz force $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$

Examine this equation closely, we realize that the three components $\vec{F}$, $\vec{E}$, and $\vec{v} \times \vec{B}$ are either all axial vectors (pseudovector) or polar vectors (ordinary vector).

To give a lightning review, axial and polar vectors transform in the same manner under the rotation when the determinant of the rotation matrix is 1; an extra minus sign emerges when the axial vector is under a transformation such as reflection and inversion, where the determinant of the transformation is -1. Now, to sum it up, the relation between axial and polar vectors is:

$axial \times polar = polar$

$axial \times axial = axial$

$polar \times polar = axial$

Well, we know that $\vec{F} = \frac{d}{dt}(m\vec{v})$ where $\vec{v} = \frac{d}{dt}\vec{r}$, so $\vec{F}$ is obviously a polar vector, and so is $\vec{v} \times \vec{B}$ and $\vec{E}$. By the rules we sum up above, we see that $\vec{B}$ is then an axial vector.

Now we guess the equation of motion for the $\vec{E}$ and $\vec{B}$. We have the superposition of forces and fields, so we will start with the simplest linear form, exploiting our intuition that the universe should be simple. Following this, the most reasonable initial guess is $\frac{d\vec{E}}{dt} \propto \vec{E}$ and $\frac{d\vec{B}}{dt} \propto \vec{B}$.

However, an unphysical outcome of this relation is that this leads to exponential growth or decay of the fields over time. If the Coulomb force $q\vec{E}$ just grow or decay exponentially over time, we will either be crumbled up into a small piece of flesh or explode and get pieces of us flying off in all directions. Too much bloodiness; let us reject this initial guess.

So what if we swap up the variables and get $\frac{d\vec{E}}{dt} \propto \vec{B}$ and $\frac{d\vec{B}}{dt} \propto \vec{E}$.

This indeed solve our previous concern, but new issues emerge. We are mixing up axial and polar vectors now. Ok let us massage our expressions a bit more $\frac{d\vec{E}}{dt} \propto \vec{r} \times \vec{B}$ and $\frac{d\vec{B}}{dt} \propto \vec{r} \times \vec{E}$

Good. We have new issues now. We ask ourselves, do we believe that the physics of these fields should looks the same under a constant translation $\vec{r} \to \vec{r} + \vec{c}$? The right answer is yes. Then the above speculation again fail us, so instead of using $\vec{r}$, we will use $\boldsymbol{\nabla} \equiv \partial r_i$ which is both a polar vector and invariant under a constant translation. And because $\boldsymbol{\nabla} \cdot \vec{E}$ and $\boldsymbol{\nabla} \cdot \vec{B}$ result in scalars, so we choose $\boldsymbol{\nabla} \times \vec{E}$ and $\boldsymbol{\nabla} \times \vec{B}$. With up to a constant, we fix the forms for the equation of motion of the two fields:

$$\frac{\partial \vec{E}}{\partial t} = c_1 \boldsymbol{\nabla} \times \vec{B}, \qquad \frac{\partial \vec{B}}{\partial t} = c_2 \boldsymbol{\nabla} \times \vec{E} \tag{3}$$

Now imagine at $t = 0$ an isolated neutral box with a thin sheet separating the two sides and at $t = dt$ the right side is all positive and left side all negative (Figure 1). By conservation of charge, there must be some current $\vec{j}$ that flows through the sheet between $t = 0$ and $t = dt$. With some experimental input, we know that $\vec{B} = 0$ at $t = dt$ and there is a uniform electric field pointing from right to left, so the first reasonable thought would be to relate these two changes and write down $\frac{\partial \vec{E}}{\partial t} = c_3 \vec{j}$

Combine this and our previous guess, we have

$$\frac{\partial \vec{E}}{\partial t} = c_1 \boldsymbol{\nabla} \times \vec{B} + c_3 \vec{j}, \qquad \frac{\partial \vec{B}}{\partial t} = c_2 \boldsymbol{\nabla} \times \vec{E} \tag{4}$$

To get the source equation for the above two, take the divergence on both sides and use the continuity equation $\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \vec{j} = 0$. Finally, know that divergence of curl is always 0, we get

$$\boldsymbol{\nabla} \cdot \vec{E} = -c_3 \rho, \qquad \boldsymbol{\nabla} \cdot \vec{B} = 0 \tag{5}$$

And we have the full set of the Maxwell Equation, and the constant coefficient are to be determined by choice of unit. Then we start to worry,

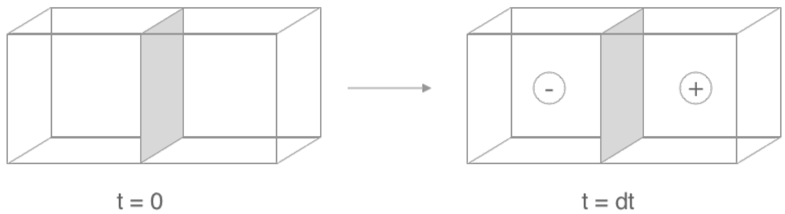

Figure 1: Left: two neutral boxes with a thin sheet in between at $t = 0$; right: reveal the charges in two boxes at $t = dt$

the time derivative is not squared. But worry not, $\vec{E}$ and $\vec{B}$ do not exist by themselves, as can be seem directly from their equation of motion. So let us combine the two into one equation of motion. Take away charge distribution and current to obtain a vacuum for the propagation of our two fields. Take another partial time derivative of one of the equation of motion to obtain

$$\frac{\partial^2 \vec{E}}{\partial t^2} = c_1 \boldsymbol{\nabla} \times \left(\frac{\partial \vec{B}}{\partial t} \right) = c_1 c_2 \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \vec{E} = -c_1 c_2 \nabla^2 \vec{E} \tag{6}$$

The same is for $\vec{B}$, and we end up with a reversible process. We know that light follows the same path as time is reversed, so this is no surprise.

3 Entropy and Planck

After the detour into electromagnetism, we come back to entropy. To bridge thermal physics and statistical mechanics, we need the brilliance of Planck. As we already discussed the entropy as a measure of preference for different states of a system, it is not hard to related it to probability. For two independent subsystem within a system, let the probability (or equivalently, multiplicity) each be W_1 and W_2, so the total probability is $W = W_1 W_2$. We speculate that entropy is some function of the probability, and we add the entropy of the two subsystem to get total entropy $f(W) = f(W_1 W_2) = f(W_1) + f(W_2)$. Take partial derivative with respect to W_1 on both sides and then take partial derivative with respect to W_2:

$$f''(W_1W_2)W_1W_2 + f'(W_1W_2) = f''(W_1)\frac{\partial W_1}{\partial W_2} + f''(W_2)\frac{\partial W_2}{\partial W_1} \tag{7}$$

The right side vanishes, and we have an ordinary differential equation:

$$f''(W)W + f'(W) = 0 \tag{8}$$

It has the general solution $f(W) = c_1 \ln W + c_2$. In this way, Planck figured out c_1 to be the Boltzmann constant. However, if we are more logical and let the system have 0 entropy when it is completely classical with $W = 1$, and without loss of generality let $c_2 = 0$, then we can completely ditch the Boltzmann constant and write $f(W) = S = \ln W$. This expression is completely equivalent to the thermal physics expression of entropy, and will serve an important role in our following discussion of information.

4 Entropy and Quantum Computing

Let us first re-visit the Maxwell's demon thought experiment. We have an isolated box with a bunch of gas molecules in equilibrium randomly distributed in the box. Now we place a wall in the box to separate the left and right side of the box. On the wall we have a small door guarded by the cunning demon. The demon will observe the speed of the molecules on the left side closely and will open the door when a fast molecule is rushing towards it. After a long enough period of time, the left side will become cold and the right side will become hot. With this process, we are transferring heat from a cold object to a hot object in an isolated system, in obvious violation of the second law of thermodynamics.

So what should we do? The resolution is in the entropy. Because the demon is a finite physical system, so it must have a finite memory, meaning that its composition must have a finite arrangements to store the information of the molecules. Suppose the demon is composed of n two-level system, then it can have at most 2^n arrangements, and it can only register at most n bits of information. At some point of time, the demon has to forget information in order to register new information of the new molecules that it wants to let through.

Now pause and ponder, what is information? It is some specific order of a given amount of objects. In a way, we could say that information is the physics, the rule, and the way the universe operate. And that life is also some kind of information and order. But it is fragile like a straight flush in poker; if you re-shuffle it, it is no longer there. But information is also quite indomitable. There is no way you can destroy information with no cost.

In fact, whenever we erase one bit of information, we are releasing the order and generating a corresponding entropy of the amount $S = \ln 2$, and then we can relate this to the theoretically minimum amount of energy required to release this much entropy by $\Delta E = T\Delta S$, given that pressure and volume of the concerned system are hold constant.

So we have resolved the Maxwell's demon paradox. When the demon runs out of memory, it has to erase some information, and that is when it has to pay the energy bill it owes.

This connection between minimum energy required and the erasure of information was known as Landauer's Principle. And it gave rise to the importance of quantum computing.

By Landauer's principle we know something about the classical computation. Whenever we do a logic gate computation like the AND gate or the OR gate which requires two inputing bits but spits out only one output bit, we are prone to the erasure of one bit of information, and energy cost of $\Delta E = T \ln 2$. And after n operations, energy cost is $\Delta E \sim nT \ln 2$.

Then we wonder, what about the reversible computations? When $\Delta S = 0$, there should be no required energy cost. Quantum computing is then the answer. Ignoring the non-conservative forces such as friction, we know that classical dynamics is deterministic and reversible, and quantum is the same thing minus the deterministic part. This feature of quantum has another name: unitarity. It is easy to understand. All quantum gates can be represented by unitary matrices, so the reverse is just the inverse of the matrices, and no quantum bits get erased. No entropy involved. No energy bill to pay!

5 Computational Power of a Black Hole and of the Universe

We see that entropy is in the heart of both the matter side of the universe, and the information side of the universe. Now we are this far into the connection between entropy and the physics of computation, we are going to push even further.

5.1 Bekenstein Bound

Recall that, for a black hole, the entropy is distributed on the event horizon $\sim L^2$ unlike ordinary entropy that we know. So we take a guess and promote the entropy to a generalized entropy $S_{gen} = S_{BH} + S_{out}$ where S_{out} is the entropy of ordinary matter going out of the black hole. We insist on this generalized entropy, and get the generalized second law of thermodynamics:

$$dS_{gen} \geq 0 \tag{9}$$

We know that the entropy of the black hole is:

$$S \sim \frac{A}{l_p^2} \tag{10}$$

where area $A \sim (\frac{GM}{c^3})^2$ and Planck length $l_p^2 \sim \frac{G\hbar}{c^3}$. We know that the Schwarzschild radius is $R \sim \frac{GM}{c^2}$. Now we are going to dive into the god-given units $c = \hbar = G = k = 1$ for the sake of dimension analysis. We would like to extremize the entropy to obtain an upper bound for the S_m, so we can vary the area of the black hole $\delta A \sim M\delta M$, and by dimension analysis we see that $\delta M \sim \frac{ER}{l}$ where l is some length, but we are not too concerned about it. We will see why.

$$\delta S_{BH} = \frac{dS_{BH}}{dM}\delta M \sim \frac{M}{l}ER \tag{11}$$

We see that $\frac{M}{l}$ is unity and so is ER. So which part shall we keep? Remember that we are concerned about the upper bound of entropy of arbitrary thing outside of the black hole, which need not be related to the mass of the black hole, but certainly has to relate to its energy, which connects directly to our concerned entropy bound. So we will consider $\frac{M}{l}$ no more and write $\delta S_{BH} \lesssim ER$. This is in fact known as the Bekenstein bound. The genius of Bekenstein speculated the generalized entropy and figured everything out.

Now apply the generalized second law with the knowledge that $S_{out} = -S_{matter}$ and $dS_{gen} = \delta S_{BH} - S_{matter} \geq 0$ we get an upper bound for entropy of matter $S_{matter} \leq \delta S_{BH} \lesssim ER = \frac{ER}{\hbar c}$. Well, indeed we see that, if there exists an ultimate matter that saturated the entropy bound, it would be black hole. So we shall take a look at how it would perform.

5.2 Computational Power

But first, we need to understand the physics of computational power a bit better. We will see that there are two limits imposed by physics. Energy limits the speed, and entropy limits the memory accessible for computation.

Let us derive our first limitation. Maximally how many operations can a matter perform in a unit time? Well, we should ask out old friend Heisenberg: $\Delta x \Delta p \sim \hbar$, which subsequently give us $\Delta E \Delta t \sim \hbar$. So we obtain a limit on the speed of operation:

$$\frac{1}{t} \sim \frac{E}{\hbar} \tag{12}$$

Then we know that information are expressed in terms of bits, so $I = \log_2 N$ where N is the total number of accessible states. In terms of entropy $S = \ln W$, W is the total number of physically accessible states, and can connect directly to N since all computational units are essentially physical matter. And we can express the total number of bits a physical system can ultimately have:

$$I = \frac{S}{\ln 2} \tag{13}$$

Merge the two limitations, we have the number of operations per second per bit a system can perform: $\frac{E}{\hbar}\frac{\ln 2}{S}$. Holding volume constant, we have $\frac{T \ln 2}{\hbar}$. And we see that temperature of the system determines solely the speed of operation on each bit.

5.3 Computing with Black Hole

Now we are ready to think about a black hole. Can a black hole be programmed? Well, as Hawking and Bekenstein indicated, black hole also have entropy, and that information is not destroyed when it falls into the black hole, and will be re-emitted in the form of Hawking radiation. What goes in does come out, just in an altered form. So we can form a black hole with initial condition we desire, and let it evolve in the Planckian dyanamics at the horizon, then the output will be encoded in the form of Hawking radiation. We just need to know how to decode it.

We know that the black hole has entropy $S \sim \frac{A}{l_p^2}$ where $A \sim (\frac{GM}{c^2})^2$ is the surface area of the horizon, and $l_p \sim \sqrt{\frac{G\hbar}{c^3}}$ is the Planck length. Finally, we know that the black hole is radiating at the Hawking radiation temperature $T_H \sim \frac{\hbar c^3}{GM}$, with a total life time of, by dimension analysis, $t_{BH} \sim \frac{G^2 M^3}{\hbar c^4}$.

The rest is just plug and chug. With $E \sim mc^2$, a black hole can perform $\sim \frac{mc^2}{\hbar}$ operations per unit time. It can register $\sim \frac{GM^2}{\hbar c}\frac{1}{\ln 2}$ bits in total. If we have a black hole of 1 kg, it would be $\sim 10^{34}$ operations per second and $\sim 10^{15}$ bits registered in total. Note that this number of bits is something we can achieve already, but our corresponding computational power is far from comparable to an ultimate computer like black hole.

5.4 Ultimate Computational Power of the Universe

At the end of this journey, let us find the computational power of the universe. The age of the universe is estimated to be $\sim 10^{10}$ years $\sim 10^{17}$ seconds. The volume, by which we mean the particle horizon and the boundary of all the information we can attain, of the universe is $\sim (ct)^3$ since the big bang. Then we can easily see that the number of operations perform by the universe is $\sim \frac{(ct)^3\rho c^2}{\hbar} = \frac{\rho c^5 t^4}{\hbar}$.

The rest of the work is to find the density ρ of the universe. Let us imagine. There must exists some critical density ρ_c such that, if we go below this value, then the kinetic energy will win, and the universe will expand forever; if we go beyond this value, then the gravitational energy will win, and the universe will eventually contract. Well, with the opposition and compromise method, we can easily determine the form of the density.

For galaxies at distance R , it is moving away at the speed of HR where $H \sim \frac{1}{t}$ is the Hubble constant. So the kinetic energy is $\sim \frac{1}{2}m(HR)^2$. And the gravitational energy of the galaxies is $\frac{G(\frac{4\pi}{3}R^3\rho)m}{R} = \frac{4\pi}{3}GmR^2\rho$. Equating kinetic and gravitational energy, we have:

$$\rho = \frac{3}{8\pi}\frac{H^2}{G} \sim \frac{1}{Gt^2} \tag{14}$$

So the speed at which the universe can perform is $\sim \frac{1}{Gt^2}\frac{c^5t^4}{\hbar} = \frac{t^2}{t_p^2}$ where t_p is the Planck time. This is $\sim 10^{120}$ operations that the universe can compute over the course of its life time. Quite impressive. Let us shine some more light on this. Since number of operations is a dimensionless number, of course we are expecting $f(\frac{t}{\tau})$ where τ must be a combination of some fundamental constants c, G, and $\hbar$, and the only one that makes the most sense is the Planck time. $f(\frac{t}{\tau})$ must be in the form of some polynomial, but since we know that the horizon of the universe expand in square term, we are expecting $\frac{t^2}{t_p^2}$ term to be our answer. And that's it!

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