Laurent Series and Residue Theorem

Yiluo Li

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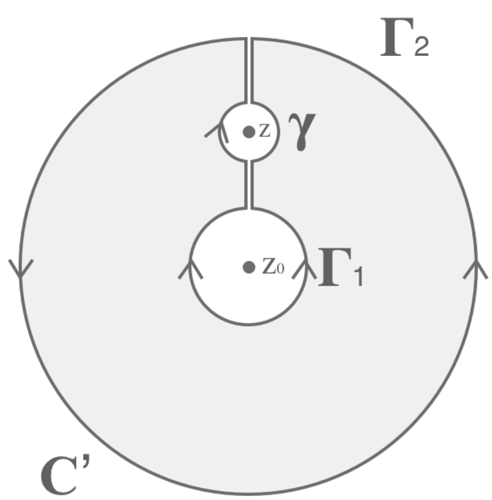
1 Laurent Series

FIrst, let us consider the following setup:



We have a function $f(z)$ which has a singularity at z_0. The shaded region is simply connected region with contour C'. By Cauchy Theorem, we know that the contour integral around C' is 0, because the shaded region is analytic:

$$\oint_{C'} \frac{f(z')dz'}{z' - z} = 0 \tag{1}$$

Here we need to make sure we understand z, z_0, and z'.

1. z is an arbitrary point between Γ_1 and Γ_2;
2. z' is on the contour C' as we can see from the integral;
3. z_0 is the singularity that we are about to expand our $f(z)$ around.

Now we can rewrite Equation 1 as a combination of path integrals along Γ_1, Γ_2 and γ:

$$\oint_{\Gamma_1} \frac{f(z')dz'}{z'-z} - \oint_{\gamma} \frac{f(z')dz'}{z'-z} - \oint_{\Gamma_2} \frac{f(z')dz'}{z'-z} = 0 \tag{2}$$

By Cauchy Integral Theorem, $f(z) = \frac{1}{2\pi i}\oint_{\gamma} \frac{f(z')dz'}{z'-z}$, therefore:

$$f(z) = \frac{1}{2\pi i}\left(\oint_{\Gamma_1} \frac{f(z')dz'}{z'-z} - \oint_{\Gamma_2} \frac{f(z')dz'}{z'-z}\right) \tag{3}$$

$$= \frac{1}{2\pi i}\left(\oint_{\Gamma_1} \frac{f(z')dz'}{(z'-z_0)-(z-z_0)} - \oint_{\Gamma_2} \frac{f(z')dz'}{(z'-z_0)-(z-z_0)}\right) \tag{4}$$

On the Γ_1 circle, $|z'-z_0| > |z-z_0|$, and on the Γ_2 circle, $|z'-z_0| < |z-z_0|$, so we can rewrite the above to:

$$f(z) = \frac{1}{2\pi i}\left(\oint_{\Gamma_1} \frac{1}{z'-z_0}\frac{f(z')dz'}{1-\frac{z-z_0}{z'-z_0}} + \oint_{\Gamma_2} \frac{1}{z-z_0}\frac{f(z')dz'}{1-\frac{z'-z_0}{z-z_0}}\right) \tag{5}$$

$$= \frac{1}{2\pi i}\left(\oint_{\Gamma_1} \frac{f(z')dz'}{z'-z_0}\sum_{n=0}^{\infty}(\frac{z-z_0}{z'-z_0})^n + \oint_{\Gamma_2} \frac{f(z')dz'}{z-z_0}\sum_{n=0}^{\infty}(\frac{z'-z_0}{z-z_0})^n\right) \tag{6}$$

In the integral with respect to dz', $(z-z_0)^n$ is just constant, so we can take it out:

$$\frac{1}{2\pi i}\left(\sum_{n=0}^{\infty}(z-z_0)^n \oint_{\Gamma_1} \frac{f(z')dz'}{(z'-z_0)^{n+1}} + \sum_{n=0}^{\infty}\frac{1}{(z-z_0)^{n+1}}\oint_{\Gamma_2} f(z')(z'-z_0)dz'\right) \tag{7}$$

Re-index the second summation and let it run from $-\infty$ to -1:

$$\frac{1}{2\pi i}\left(\sum_{n=0}^{\infty}(z-z_0)^n \oint_{\Gamma_1} \frac{f(z')dz'}{(z'-z_0)^{n+1}} + \sum_{n=-\infty}^{-1}(z-z_0)^n \oint_{\Gamma_2} \frac{f(z')dz'}{(z'-z_0)^{n+1}}\right) \tag{8}$$

Now combine the two terms:

$$f(z) = \frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} (z - z_0)^n \oint_C \frac{f(z')dz'}{(z' - z_0)^{n+1}} \tag{9}$$

where C is a contour between Γ_1 and Γ_2.

2 n^{th} Order Derivative

In this section, we will show that if a function $f(z)$ has a first derivative at $z = z_0$, then it has derivative of n^{th} order $\forall n \in \mathbb{N}$.

$$f(z) = \frac{1}{2\pi i} \oint \frac{f(w)dw}{w - z} \tag{10}$$

Then the derivative of $f(z)$ evaluated at $z = z_0$ is:

$$\begin{aligned}
\left(\frac{df}{dz}\right)_{z=z_0} &= \lim_{z\to z_0} \left(\frac{f(z) - f(z_0)}{z - z_0}\right) && (11)\\
&= \lim_{z\to z_0} \frac{1}{2\pi i} \frac{1}{z - z_0} \left(\oint \frac{f(w)dw}{w - z} - \oint \frac{f(w)dw}{w - z_0}\right) && (12)\\
&= \lim_{z\to z_0} \frac{1}{2\pi i} \frac{1}{z - z_0} \left(\oint dw f(w)(\frac{1}{w - z} - \frac{1}{w - z_0})\right) && (13)\\
&= \lim_{z\to z_0} \frac{1}{2\pi i} \frac{1}{z - z_0} \left(\oint dw f(w)(\frac{z - z_0}{(w - z)(w - z_0)})\right) && (14)\\
&= \lim_{z\to z_0} \frac{1}{2\pi i} \oint \frac{f(w)dw}{(w - z)(w - z_0)} && (15)\\
&= \frac{1}{2\pi i} \oint \frac{f(w)dw}{(w - z_0)^2} && (16)
\end{aligned}$$

Do this repeatedly for higher order derivatives, we will get a list of derivatives:

$$\begin{aligned}
\frac{df}{dz} &= \frac{1}{2\pi i} \oint \frac{f(w)dw}{(w - z)^2} && (17)\\
\frac{d^2 f}{dz^2} &= \frac{2}{2\pi i} \oint \frac{f(w)dw}{(w - z)^3} && (18)\\
f^{(n)}(z) &= \frac{n!}{2\pi i} \oint \frac{f(w)dw}{(w - z)^{n+1}} && (19)
\end{aligned}$$

Swapping variables:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint \frac{f(z)dz}{(z-z_0)^{n+1}} \tag{20}$$

And we see that $f(z)$ has derivatives of all orders. This equation will also be used when we deal with the Residue Theorem in the next section.

3 Residue Theorem

Residue is defined as:

$$\sum_j \operatorname{Res} f(z, z_j) \equiv \frac{1}{2\pi i} \oint f(z)dz \tag{21}$$

where z_j are the singularities. Now, for a pole at z_0 with order n, it has the form:

$$f(z) = \frac{\varphi(z)}{(z-z_0)^n} \tag{22}$$

where $\varphi(z)$ is analytic. Then from the definition of residue:

$$2\pi i \operatorname{Res} f(z_0) = \oint f(z)dz = \oint \frac{\varphi(z)}{(z-z_0)^n} dz \tag{23}$$

Recall from Equation 20:

$$\oint \frac{f(z)dz}{(z-z_0)^{n+1}} = f^{(n)}(z_0)\frac{2\pi i}{n!} \tag{24}$$

Plug into Equation 23 we get:

$$2\pi i \operatorname{Res} f(z_0) = \varphi^{(n-1)}(z_0)\frac{2\pi i}{(n-1)!} \tag{25}$$

$$\operatorname{Res} f(z_0) = \frac{1}{(n-1)!} \lim_{z\to z_0} (\frac{d}{dz})^{(n-1)} \left((z-z_0)^n f(z)\right) \tag{26}$$

where in the last equality we use the fact that $\left(\frac{d\varphi}{dz}\right)_{z=z_0} = \lim_{z\to z_0} \frac{d}{dz}\varphi(z)$ and that $\varphi(z) = (z-z_0)^n f(z)$ which we get from the beginning of this section. This is the residue for $f(z = z_0)$ where z_0 is a pole of order n.
Now for the last step, plug in our Laurent expansion Equation 9 for $f(z)$ in

this residue formula, which you should do by looking at the equation, but not actually plugging in the expansion because it is so darn long. We know that, for Laurent expansion around a pole, the principle part, $\frac{1}{(z-z_0)^m}$ terms where $m > 0$, of the series will go to at most $m = n$, $\frac{1}{(z-z_0)^n}$ where n is the order of the pole (well that is definition of a pole of order n).

Think about this last step carefully: after you plug in the Laurent series expanded around a pole at z_0 of order n, only the $\frac{1}{z-z_0}$ term in the series will survive; the rest of the principle part of the series will vanish due to the derivative, and the Taylor series part, $(z - z_0)^m$ terms where $m > 0$, will vanish due to the limit $z \to z_0$. Therefore, we see that the residue of a pole is always the coefficient of the $\frac{1}{z-z_0}$ term in the Laurent series expanded around $z = z_0$.