Hydrodynamics of Accretion Disks in High Energy Astrophysics

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1 Introduction

There is little doubt that the universe is our most natural lab. Deep into it, we have high energy sources cranking out some of the most impossible experiments, for free. So let's study those before we build another LHC.

In this paper, we will be examining the relevant physics of the accretion disks, arguably the most powerful source of energy. It is generally a subject of high energy astrophysics, so we are looking at the energy scales from $0.1\,\mathrm{keV}$ to $10\,\mathrm{TeV}$, corresponding to X-ray and γ-ray photons.

First off, let us estimate the amount of energies released from different sources. In chemical reaction, we have $\sim 10\,\mathrm{eV}$ per atom involved , which is evident from recalling the ground state energy of hydrogen. This, in CGS, is $\sim 10^{12}\,\mathrm{erg\,g^{-1}}$, with an efficiency of $\eta \approx 10^{-9}$ per unit rest mass energy.

Another source is nuclear reaction, which is of order $\sim 10\,\mathrm{MeV}$ per nucleus involved, roughly the amount fusing a hydrogen into a helium. This is $10^{19}\,\mathrm{erg\,g^{-1}}$ with efficiency $\eta \approx 10^{-2}$, many orders more efficient than chemical reaction.

Our main interest is gravitational free fall. Energy per unit mass released during accretion onto our Sun is:

$$E \sim \frac{GM_\odot}{R_\odot} \sim 10^{15}\,\mathrm{erg\,g^{-1}} \tag{1}$$

with an efficiency $\eta \approx 10^{-2}$. So to accrete the entire mass of Sun onto itself, it will release energy of order $4 \times 10^{48}\,\mathrm{erg}$. Divide this by the solar luminosity, we see that gravitational energy can power the Sun for $\sim 10^7\,\mathrm{yrs}$, which is much less than the age of the Solar system $\sim 4.5 \times 10^9\,\mathrm{yrs}$, so clearly gravitational energy is not the dominant source powering our Sun.

Chemical reaction is also hopeless in its efficiency. So we are left with nuclear reaction, which indeed is the dominant source of power for our Sun, and we have observed Solar neutrinos as a direct evidence to that claim.

However, just examine Equation 1, we see that if we have a supermassive object, small radius object in the center of accretion, then the efficiency might surpass that of nuclear reaction. No problem, neutron stars and black holes are perfect examples in this case. March onto gravitational accretion!

1.1 One Note on Notation

Here we are using the CGS system, which is common to astrophysics. But of course, in the fly by night spirit, natural units still makes the most sense. In the end, aliens don't care about pesky conversion factors. Physics is physics.

2 First Thought on Eddington Luminosity

2.1 Opacity

Before we go into any discussion, we will necessarily need to learn to talk in the astrophysicists' tongue a bit. In astrophysics, an important concept is opacity. It comes from the radiative transfer equation, a very important equation for astrophysicists:

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = j_\nu - \alpha_\nu I_\nu \tag{2}$$

where I_ν is the intensity of light at some frequency ν, s is the length along the path of the light, j_ν describes the source of radiation, and α_ν is the coefficient of absorption, so the term $\alpha_\nu I_\nu$ describes how photons are absorbed by medium along its path of propagation. To get a bit more intuition of it, consider a situation with no emission, so $j_\nu = 0$, then the solution to this differential equation is just

$$I_\nu = I_\nu(0)e^{-\int \alpha_\nu ds} \tag{3}$$

which is constant attenuation with no emission from the medium. Let's understand α_ν. The relevant parameter concerning attenuation in the medium is how much matter (which is essentially just hydrogen in astrophysics) and the interaction strength between photon and electron per electron involved. Photon-proton interaction is weaker by square the ratio of electron and proton mass, and we won't consider it here. So first, we have density ρ of the medium. Think about it physically, if we have more matter, of course there will be more attenuation, so α must be linearly (or higher order, but we don't care) related to ρ. Then by dimensional analysis, α has $\frac{1}{L}$ so the other parameter must be $\frac{L^2}{M}$. We give it a variable, κ_ν, and call it opacity. It describes the cross section over mass per electron involved. Since we mostly have hydrogen, this is usually proton mass.

Some more jargons to throw around, we call $\tau_\nu = \int \alpha_\nu ds$ optical depth. When $\tau_\nu > 1$, we say the medium is optically thick; when $\tau_\nu < 1$, we say the medium is optically thin. Look at our expression for I_ν above, this is really just a statement about identifying that exponential cut off point for photon intensity along the path in the medium.

2.2 Eddington Luminosity

Now we will talk about accretion. We start simple. Consider a non-relativistic, spherically symmetric, optically thin medium accreting onto mass M at the origin (see Figure 1)

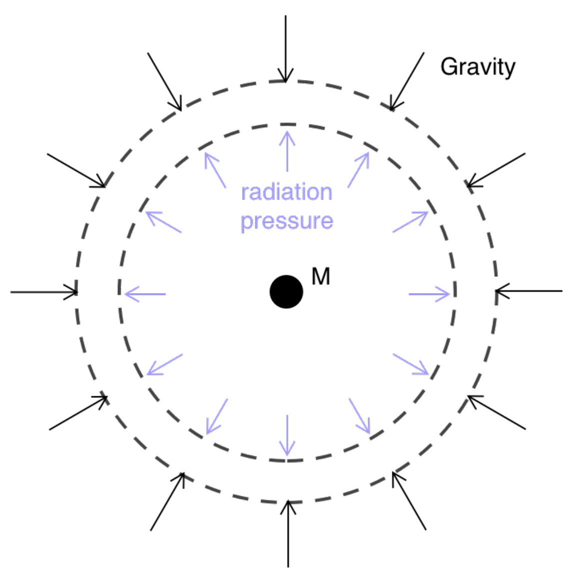

Figure 1: Radiation pressure balances the gravitational pull

Also, let us only consider photon energies less than $m_e c^2$, then we have Thomson scattering, and the opacity is $\kappa_T = 0.4\,\mathrm{cm}^2\,\mathrm{g}^{-1}$ (or $0.35\,\mathrm{cm}^2\,\mathrm{g}^{-1}$ if we want to take into the second abundant element, helium, into account). Now, since we assumed optically thin disk, photons can travel outward to infinity at constant speed of light, and its outward radiative acceleration (radiative pressure force per unit mass) at a distance r can be obtained by:

$$F = \frac{L}{4\pi r^2 c}\kappa_T \tag{4}$$

where the $4\pi r^2$ is just there to account for the spherical symmetry, and L is the luminosity. Quick dimensional analysis reveals that $\frac{L}{4\pi r^2 c}$ can also be interpreted as momentum flux.

The other force within our consideration is gravity, so equating the two, we can find a critical value in luminosity, beyond which gravity defeats radiation and everything gets dragged in.

$$\frac{GM}{r^2} = \frac{L}{4\pi r^2 c}\kappa_T \tag{5}$$

Almost magically the r^2 cancels out, so the critical value of the luminosity depends linearly on the mass at the origin where accretion happens. But of course, we know this is not magic. This is just because both gravity and the accretion are spherically symmetric in our consideration. And from this analysis, Eddington luminosity is defined through

$$L_E = \frac{GM}{4\pi\kappa_T}c \tag{6}$$

Nature seems to know about the Eddington luminosity. We can see in Figure 2 that there seem to be an upper limit imposed by the Eddington luminosity. However, there are also multiple exceptions to this rule, which is an ongoing confusion for the astrophysical community.

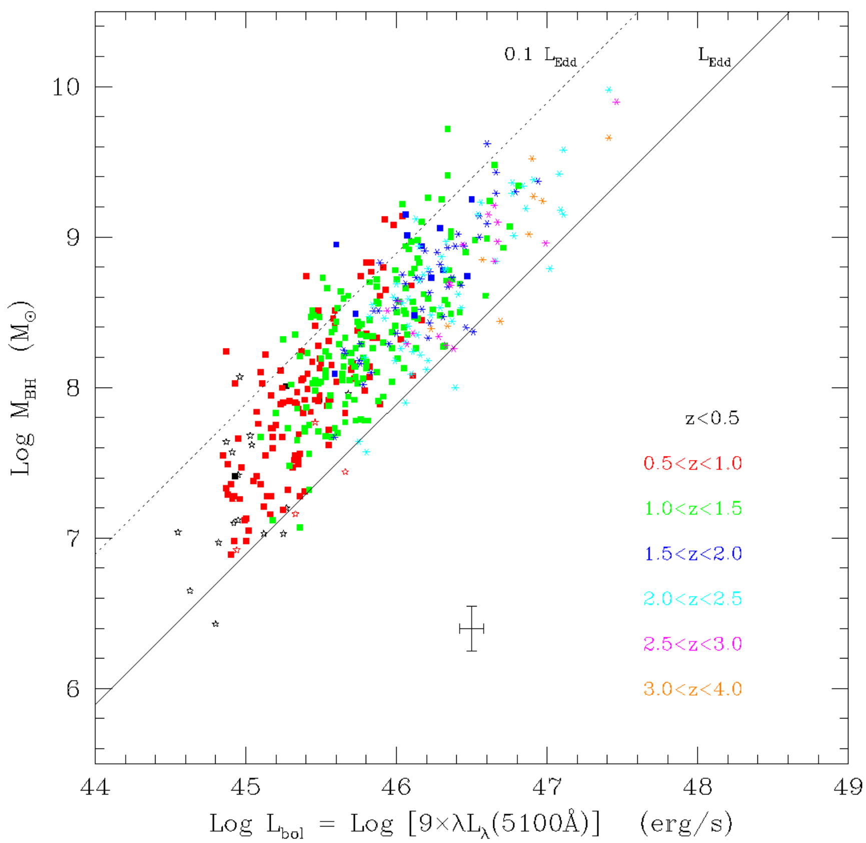

Figure 2: Relation between mass and luminosity of black holes observed.

3 Hydrodynamics Approach - Bondi Accretion

3.1 Simple Model

Now we consider ourselves graduated from the baby question. Let's consider a child problem, a spherically symmetric, adiabatic hydrodynamics accretion, because we understand that the inflow of matter is more like a fluid flow than "hydrostatic" setup in the baby problem. This hydrodynamics process is also known as Bondi accretion. But of course this child problem is still boring for any observer. It is adiabatic, so no one will see any light. However, this problem is relevant for cases like accretion onto supermassive black holes in the center of our galaxy or accretion of stellar wind onto compact objects like neutron stars. And if we build on it, we can incorporate angular momentum and shear stress for a more realistic view.

This model is also significant since it is the simplest one for a flow with a critical point, and such mathematical behavior is relevant to many other celestial structures in astrophysics.

We will consider some mass point sitting at the origin in an infinite medium of matter that is distributed uniformly at infinity, denoted by ρ_∞, with uniform pressure P_∞, and hence constant sound speed $c_{s\infty}$. As usual, we write down the continuity equation first:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \vec{v}) = 0 \tag{7}$$

Note that we ignore the angular motion for now, so there is no viscosity and angular velocity involved, $\vec{v} = -v\hat{r}$. Also we assume this is a stationary flow, so density at fixed radii is constant over time, and we can re-write out continuity equation into spherical coordinate:

$$\frac{1}{r^2}\frac{\mathrm{d}r^2\rho v}{\mathrm{d}r} = 0 \tag{8}$$

$$4\pi r^2 \rho v = \dot{M} = const \tag{9}$$

We get accretion rate is constant at all radii. Then let us write down the Euler's Equation, which I like to refer to as simply conservation of momentum:

$$\rho\frac{\partial \vec{v}}{\partial t} + \rho(\vec{v} \cdot \boldsymbol{\nabla})\vec{v} = -\boldsymbol{\nabla} P - \rho \boldsymbol{\nabla} \Phi \tag{10}$$

Here Φ is just the gravitational potential $\Phi = -\frac{GM}{r}$. Again, write everything into spherical coordinate:

$$v\frac{\mathrm{d}v}{\mathrm{d}r} + \frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} + \frac{GM}{r^2} = 0 \tag{11}$$

The $\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r}$ term is begging us to take care of it in terms of the speed of sound and get rid of the notion of pressure, so let's do it. Recall that this is an adiabatic flow, so the material derivative (total derivative) of the adiabatic flow $P\rho^{-\gamma}$ should vanish, where γ is the adiabatic index in terms of specific heat capacities, and we will assume it constant for simplicity.

$$P\rho^{-\gamma} = const = P_\infty \rho_\infty^{-\gamma} \tag{12}$$

We know that the adiabatic speed of sound is defined to be $c_s^2 = \frac{\gamma P}{\rho}$. Simply use these two facts and plug into the $\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r}$ term:

$$\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} = \frac{P_\infty \rho_\infty^{-\gamma}}{\rho}\frac{\mathrm{d}\rho^\gamma}{\mathrm{d}r} \tag{13}$$

Simplify this in favor of speed of sound, we will eventually get:

$$\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} = \frac{\mathrm{d}}{\mathrm{d}r}\frac{c_s^2}{\gamma-1} \tag{14}$$

Put this back into the conservation of momentum equation, and write everything to be a total derivative

$$\frac{\mathrm{d}}{\mathrm{d}r}(\frac{1}{2}v^2 + \frac{c_s^2}{\gamma-1} - \frac{GM}{r}) = 0 \tag{15}$$

$$\frac{1}{2}v^2 + \frac{c_s^2}{\gamma-1} - \frac{GM}{r} = const = \frac{c_{s\infty}^2}{\gamma-1} \tag{16}$$

where in the last equality we use the fact that velocity of the flow and effect of gravity both vanishes as $r \to \infty$. This equation should not surprise people. If you look close enough, you should realize that this is just the Bernoulli equation for compressible fluid (taken care of by the introduction of sound speed and adiabatic index).

At this point, we bump into some issues. We have 3 algebraic equations with 5 vairables ($c_{s\infty}$, ρ_∞, γ, M, and $\dot{M}$). Either we go and try to find more equations, or we should think about what went wrong. But of course, with some experience with point mass at the origin problem, we immediately know we should be careful about radial derivative and point mass at the origin. In the previous two equations that end up being constants:

$$\frac{\mathrm{d}}{\mathrm{d}r}(r^2\rho v) = 0 \tag{17}$$

$$r^2 v\frac{\mathrm{d}\rho}{\mathrm{d}r} + r^2\rho\frac{\mathrm{d}v}{\mathrm{d}r} = -2r\rho v \tag{18}$$

And do the similar thing to the other one

$$\frac{c_s^2}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r} + v\frac{\mathrm{d}v}{\mathrm{d}r} = -\frac{GM}{r^2} \tag{19}$$

Then we can shuffle the variables around to solve algebraically for $\frac{\mathrm{d}\rho}{\mathrm{d}v}$ and $\frac{\mathrm{d}v}{\mathrm{d}r}$.

$$\frac{\mathrm{d}\rho}{\mathrm{d}r} = \frac{\rho(GM - 2rv^2)}{r^2(v^2 - c_s^2)} \tag{20}$$

$$\frac{\mathrm{d}v}{\mathrm{d}r} = \frac{v(2rc_s^2 - GM)}{r^2(v^2 - c_s^2)} \tag{21}$$

Nevermind all the mess, I would like to point your attention to a possible singularity arising from setting $v^2 = c_s^2$ in the denominator. This means if the flow is transonic, meaning that the velocity of the flow passes the sound speed, then a singularity will develop. It would be nice to consider black hole and a light signal as an analogy. For black hole, we have an event horizon, where the "in fall" speed within is larger than speed of light, so a signal cannot be sent out of the horizon. In the hydrodynamic flow we are considering, such a horizon also exists. Speed of sound is essentially the speed of communication within the flow, carried by compressible waves. If the flow velocity surpass that of sound speed within some horizon, flow outside the horizon will never "hear" from the flow inside the horizon.

Such potential singularities are called the critical points. And the one we are talking about here is called the sonic point. However, there is nothing to fear about this singularity. If we think about physically, some fluid is traveling in space, where different parts of it is traveling at different speed, and the velocity function throughout the fluid is smooth and differentiable until we hit the sonic point. Should anything dramatic happen? No! An observer will perhaps just see compressible waves in the fluid can go one way but not the other way through the sonic point. In fact, this should be one intuition for what should happen as a particle travels across the event horizon of the black hole. Nothing dramatic should happen! As we encounter the singularity at the sonic point, we sloppy physicists are just going to shut our eyes and set the derivatives to 0. Done.

Then the rest is again just plug and shuffle. Cancel out variables to solve for $\dot{M}$, we get

$$\dot{M} = \frac{4\pi\lambda_s G^2 M^2 \rho_\infty}{c_{s\infty}^3} \tag{22}$$

where λ_s is called the accretion eigenvalue. But it is roughly unitary, so we don't really care.

This solution is sometimes called the transonic solution, as the hydrodynamics flow starts subsonic and passes through the sonic point, which is apparently relevant for modeling the black hole accretion.

3.2 Collisionless Accretion

For some extra fun, let us consider a uniform, static, collisionless medium. If we think about it hard enough, we will realize collisionless is synonym for "not involved in electromagnatic interaction", so this kind of medium is commonly known as dark matter. Let us consider a black hole moving at constant velocity u through this medium. Of course, by a change of reference frame, this becomes a problem of stellar wind accreting onto a black hole sitting at the origin. Consider a collisionless particle with impact parameter b. By energy and momentum conservation of particle far away and at closest approach (at radius R, which will absorb all particles that encounter it):

$$\frac{1}{2}u^2 = \frac{1}{2}v_{max}^2 - \frac{GM}{R} \tag{23}$$

$$ub = v_{max}R \tag{24}$$

This process has a fancy name called gravitational focusing, but really we know it is just a classical scattering process. Solving for the impact parameter we get

$$b = R\sqrt{1 + \frac{c^2}{u^2}} \approx \frac{c}{u}R \tag{25}$$

So we see that if the black hole is moving very slowly, then the cross section of this gravitational focusing will be quite large. The particles will effectively "see" a larger black hole. However, if the black hole moves too fast, then particles will just "slip" by more often; the hole will move too fast and have too little time to internalize things, same for learning physics.

Now consider the case when $b >> R$, we can quickly show that the Bondi accretion is much more significant than the collisionless accretion. For a collisionless accretion we have:

$$\dot{M}_{cl} \sim b^2 u \rho_\infty \tag{26}$$

And for Bondi accretion we have, to the order of magnitude approximation:

$$\dot{M})_B \sim r_s^2 u \rho_\infty \tag{27}$$

where we recall that r_s is the sonic radius $r_s \sim \frac{GM}{c_{s\infty}^2}$. So we can see the ratio between collisionless accretion and the Bondi accretion is

$$\frac{\dot{M}_{cl}}{\dot{M}_B} \sim \frac{b^2}{r_s^2} \sim \frac{u^2}{c^2} \tag{28}$$

This is especially true for objects like black hole, where escape velocity is of order the speed of light, so $u << c$ and collisionless accretion rates are much smaller than hydrodynamics accretion rates.

On last note. Collisionless particles do not participate in electromagnetic interaction. Therefore they do not emit photons either. However, they are still massive, so through gravitational lensing, we should estimate a different mass of the black hole from collecting photons that it emits. In fact, this was one of the early evidences for the existence of the dark matter. Yet another reason for us to study astrophysics more before building a larger LHC.

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